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Trigonometry of the quantum state space, geometric phases and relative phases

Ramón Ortega and Mariano Santander

Departamento de Física Teórica, Facultad de Ciencias, Universidad de Valladolid, E-47011 Valladolid, Spain

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Abstract

A complete set of invariants for three states in the quantum space of states $\mathcal{P}$ is obtained together with a complete set of relationships linking them. This is done in a way that preserves the self-duality of $\mathcal{P}$ and leads to a clear geometric description of invariants (distances, lateral phases; Hermitian angles, angular phases; and two purely triangular phases). Some of these invariants appear here for the first time. Symplectic area (and hence the triangle geometric phase) is proportional to a 'mixed phase excess', thus extending to $\mathcal{P}$ the relation 'area-angular excess' in the real sphere. The new triangle lateral phases provide a description, intrinsic to $\mathcal{P}$, of relative phases in a superposition. This approach also provides closed expressions for the triangle holonomy associated with the usual Fubini–Study metric in $\mathcal{P}$, as well as many other expressions for similar 'loop' operators along the triangle, including closed and exact expressions for the triangle Aharonov–Anandan geometric phase.

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1. Introduction

The aim of this paper is to study trigonometry in the quantum state space $\mathcal{P}$, to provide an approach to meaningfully discuss relative phases as invariants in $\mathcal{P}$ and to relate these invariants with the known geometric phases. The study is completely intrinsic to the space $\mathcal{P}$ itself, thus proving the possibility of a complete description of *relative phases* in terms of the (projective) space of states $\mathcal{P}$ only without recourse to the Hilbert space $\mathcal{H}$.

The reasoning usually made to support the idea '$\mathcal{H}$ has phase information and $\mathcal{P}$ has not' goes as follows: even if the two vectors $|\phi\rangle$ and $e^{i\alpha}|\phi\rangle$ represent the *same* physical state, say Φ, and the two vectors $|\psi\rangle$, $e^{i\beta}|\psi\rangle$ represent another state, say Ψ, the vector sums $|\phi\rangle + |\psi\rangle$ and $e^{i\alpha}|\phi\rangle + e^{i\beta}|\psi\rangle$ represent, in general, *different* states, the latter being actually a whole *one-dimensional* family of states parametrized by a *relative phase* $e^{i(\beta-\alpha)}$. The Pancharatnam definition for 'to be in phase' is framed in terms of state *vectors*: $|\phi\rangle$ and $|\psi\rangle$ are said to be 'in phase' when the Hermitian scalar product $\langle\psi, \phi\rangle$ is real and positive [1]. Clearly, there is no possible concept of 'to be in phase' for two *physical states*, mathematically described as rays, should these be *considered in isolation*; to be observable relative phases in a superposition require to interfere with another state. All this, though completely correct, might suggest that a *complete description* of relative phases without recourse to $\mathcal{H}$ is not possible. This conclusion is, however, unwarranted.

In any homogeneous space trigonometry involves the identification of natural invariants associated with three points and the study of relations between them. Trigonometry is the first stage to the study of the geometry of any homogeneous symmetric space [2, 3]; thus this work fits into the general trend of trying to formulate quantum mechanics in geometrical language [4].

The leading idea is that the natural trigonometric quantities in $\mathbb{C}P^\infty \equiv \mathcal{P}$ should likely represent *interesting* physical quantities; this is supported in several relevant examples. First is the Anandan–Aharonov (AA) [5, 6] phase Σ associated with cyclical evolutions in $\mathcal{P}$, proportional to the symplectic area enclosed by the circuit. As we will see, Σ turns out to be proportional to a triangle invariant first introduced by Blaschke and Terheggen [7] and independently and much later by Bargmann [8]. Another invariant, the length of a curve in $\mathcal{P}$ (relative to its natural Fubini–Study (FS) Riemannian structure) has also received a physical interpretation [4]; for two states leads to a *distance* between them. The physical relevance of the Kleinian geometry of $\mathcal{P}$ has started to receive attention after the discovery of geometric phases; as the trigonometry of the complex projective space was studied by Blaschke and Terheggen more than sixty years ago, one may wonder whether perhaps geometric phases *could* have been discovered and recognized as important much earlier, should this type of translation between the mathematics already known for $\mathbb{C}P^N$ and the physics of $\mathcal{P}$ have been done at the time.

Side lengths and symplectic area by no means exhaust the invariants associated with three states. The natural trigonometric questions for a triangle in $\mathcal{P}$ determined by three states Ψ_A, Ψ_B, Ψ_C joined by FS geodesic arcs are: *how many invariants* (under the unitary group $U(\mathcal{H})$ acting on $\mathcal{P}$ by projectivization of its linear action on $\mathcal{H}$) are there associated with a pair of states, with a pair of (real FS) geodesics meeting at a state, and with the triangle itself? How many of these invariants are *essential*? What are the *relations* between them? We approach the problem of the choice of (lateral, angular and triangular) triangle invariants in $\mathcal{P}$, and we uncover a complete set of their relationships. A few of the trigonometric equations we consider are known (Blaschke–Terheggen [7, 9], Shirokov–Rosenfeld [10], Brehm [11] or Hsiang [12]), but most of our relations seem to be new, including some very simple ones [13, 14]. In particular, the symplectic area of a triangle (and hence the AA geometric phase) appears as a *(mixed) phase excess*, in a way very similar to the well-known relation between area and angular excess holding in the trigonometry of the sphere. An intrinsic description of relative phases in $\mathcal{P}$ follows from this approach.

The study of quantum trigonometry loses no generality if we restrict to the complex two-dimensional case, $\mathcal{P} \equiv \mathbb{C}\mathcal{P}^2$. Group theoretically this space is the two-dimensional (complex) Hermitian elliptic space $SU(3)/(U(1)\otimes SU(2)) \equiv U(3)/(U(1)\otimes U(2))$. This is the quantum state space of lowest possible dimension accommodating a generic triangle whose vertices are *three* linearly independent (vector) states. The two-state system, the usual example to illustrate ideas relating geometric phases, is non-generic from the viewpoint of trigonometry, as its state space is the projective complex line $\mathbb{C}P^1$, which has only room to accommodate a degenerate (complex collinear) triangle.

In a previous paper [3] we have set forth a complete and thorough study of trigonometry for all spaces in the Cayley–Klein–Dickson (CKD) family of spaces of 'complex type', whose generic member depends on three real parameters, η, κ_1, κ_2. For $\eta > 0$, nine Cayley–Klein *complex* spaces are obtained with a *Hermitian* metric of constant holomorphic curvature $4\kappa_1$ and metric signature type $+1, \kappa_2$. The quantum space of states $\mathcal{P}$ of a three-state system, $\mathbb{C}P^2$ endowed with the Kähler FS metric—or for that matter any totally geodesic two-dimensional (over $\mathbb{C}$) submanifold of any $\mathcal{P}$—is the 'elliptic' member ($\eta = 1, \kappa_1 = 1, \kappa_2 = 1$), called *Hermitian elliptic space*, of the CKD family of Hermitian spaces. All the equations obtained in [3], when particularized for the 'elliptic' values for the parameters, give the basic equations of *trigonometry in the quantum space of states* in a form that displays explicitly this self-dual character of the complex projective space $\mathbb{C}P^2$, with angular and lateral invariants playing a fully similar role to those of lengths and angles in real spherical trigonometry. In this sense this paper must be considered as a 'physical application' to quantum physics of results in [3]. We include in section 2 an alternative study of the trigonometry in $\mathcal{P}$ leading directly to the physical interpretation of some new trigonometric invariants. In section 3 we merge the previous direct approach to the particularization for $\mathbb{C}P^2$ of the generic CKD trigonometry, and we comment on the role of the *Cartan sector*, related to the Cartan subalgebra of the rank-two algebra $\mathfrak{su}(3)$ which has no analogue in the trigonometry of S^2. In section 4 we discuss the status of the superposition principle in purely projective terms and interpret the trigonometric lateral phases of the triangle as the physical *relative phases*. Explicit and exact expressions relating the AA geometric phase (here denoted by Σ) for the triangle circuit to other invariants, or exact expressions in terms of the position state vectors of the three vertices are also obtained; these extend an *infinitesimal formula* derived by Sudarshan *et al* [16]. Further physical applications are given in section 5, where a dictionary between the trigonometric and the physical languages is set up. In the appendix we collect a large number of simple trigonometric equations whose generic $(\eta; \kappa_1, \kappa_2)$-form has already been derived in [3]. As yet another application we give an explicit derivation of the two basic structures in $\mathcal{P}$ (the FS metric and the symplectic area) starting from trigonometry.

1.1. Normalization conventions, notation and some facts of $\mathbb{C}P^2$

The ray space $\mathcal{P}$ associated with the Hilbert space $\mathcal{H}$ is defined as the set of equivalence classes of vectors: $|\Psi\rangle \sim |\Phi\rangle$ if $|\Psi\rangle = \mu|\Phi\rangle$ for a non-zero complex coefficient $\mu \neq 0$. Alternatively, $\mathcal{P}$ is the set of *unitary rays*, i.e. equivalence classes of vectors in the unit sphere $S_{\mathcal{H}}$: $|\Psi\rangle \sim |\Phi\rangle$ if $|\Psi\rangle = \mu|\Phi\rangle$ for a complex unimodular factor $|\mu| = 1$. We will denote by Ψ an arbitrary quantum state, which is a point in $\mathcal{P}$ and is identified with the ray $[\Psi]$ in $\mathcal{H}$ (or with the unitary ray in $S_{\mathcal{H}}$), and by $|\Psi\rangle$ a normalized representing vector in $\mathcal{H} \equiv \mathbb{C}^3$. Both the linear Hilbert space $\mathcal{H} - \{0\}$ and the unit sphere $S_{\mathcal{H}}$ in $\mathcal{H}$ are fibre bundles over $\mathcal{P}$, with fibres $\mathbb{C}^*$ and $U(1)$; the latter is the Hopf bundle, but we will not enter into the fibre bundle language as our aim is to fully remain in $\mathcal{P}$. Note, however, that the standard connection in the fibre bundle $S_{\mathcal{H}}$ makes all state vectors in any horizontal lifting of any curve in $\mathcal{P}$ 'in phase along the curve' in the Pancharatnam sense, thus providing the link of our results with the conventional ones.

The Hermitian product in the *linear* space $\mathcal{H}$ endows the projective space $\mathcal{P}$ with a Kählerian *Hermitian metric*. This rich structure includes a *complex structure*, a (real) *Riemannian metric* and a *symplectic structure*, all closely related. From the group theoretical point of view, $\mathcal{P} \equiv \mathbb{C}\mathcal{P}^\infty$ is a homogeneous Hermitian symmetric space of the unitary group $U(\mathcal{H})$ (or $SU(\mathcal{H})$), and the two geometrical Riemannian and symplectic natural structures in $\mathcal{P}$ come from the realization of $SU(N)$ as $SO(2N) \cap Sp(2N, \mathbb{R})$. The natural *Fubini–Study (FS)* metric is given by the *real* part of the Hermitian product between tangent vectors to $\mathcal{P}$, which we normalize so that all the geodesics (which are closed and of the same length) have a total length π. With this choice 4 and 1 are the two extremal values of the non-constant sectional curvature of the FS metric; these two values are the (constant) curvature of the two types ($\mathbb{C}P^1$ type and $\mathbb{R}P^2$ type) of *totally geodesic submanifolds in* $\mathbb{C}P^2$. The submanifolds of $\mathbb{C}P^1$ type are isometric with a real sphere $S^2_{K=4}$ of radius $1/2$ and correspond to the quantum space of a two-state system. The second type are $\mathbb{R}P^2$ submanifolds, isometric with a real projective subplane $\mathbb{R}P^2$ (and locally isometric with a real sphere $S^2_{K=1}$ of radius 1; globally the isometry involves an antipodal identification in $S^2_{K=1}$); this submanifold is totally real. When a geodesic of $\mathcal{P} \equiv \mathbb{C}\mathcal{P}^2$ is seen as inside the $\mathbb{C}P^1$ type submanifolds, it can be identified as a great circle of length π in the (unique) $\mathbb{C}P^1 \equiv S^2_{K=4}$ containing that geodesic. When it is considered within one of the many submanifolds of $\mathbb{R}P^2$ type containing it, it follows from a large circle in the $S^2_{K=1}$ covering twice $\mathbb{R}P^2$, where antipodal identification reduces the total length 2π of any $S^2_{K=1}$ geodesic to π.

The *symplectic structure* is given by (minus) the *imaginary* part of the Hermitian product between tangent vectors to $\mathcal{P}$; this normalization [15] makes the symplectic area of any whole complex $\mathbb{C}P^1$ submanifold equal to π, which is also the standard Riemannian area of the sphere $S^2_{K=4} \equiv \mathbb{C}P^1$ of radius $1/2$. The symplectic area of (any domain of) the purely real submanifolds $\mathbb{R}P^2$ is identically zero. Sometimes an extra factor 2 is included in the symplectic 2-form, whence its flux over any $\mathbb{C}P^1$ would be equal to 2π; it is convenient to relate geometric phases to Chern classes [17].

2. The direct approach to the trigonometry of $\mathcal{P}$

Let us start by recalling the invariants of two states Ψ_A, Ψ_B, for which $|\Psi_A\rangle \equiv |A\rangle, |\Psi_B\rangle \equiv |B\rangle$ represent vectors in $\mathcal{H}$. Unlike the Hermitian product $\langle A, B\rangle$, the modulus $|\langle A, B\rangle|$ *is* actually a geometric invariant in $\mathcal{P}$, as eventual phase factors in $|A\rangle, |B\rangle$ cancel out, so that $|\langle A, B\rangle|$ turns out to depend only on the rays $[A], [B]$ and not on the vectors $|A\rangle, |B\rangle$ themselves. This $|\langle A, B\rangle|$ is the *unique* $\mathcal{P}$-invariant associated with the pair of states Ψ_A, Ψ_B: the Hermitian elliptic space is a *rank one* space, and any other invariant of the pair Ψ_A, Ψ_B can be expressed in terms of $|\langle A, B\rangle|$. Geometrically, the invariant $|\langle A, B\rangle|^2$ is related to the *Fubini–Study distance* c between the two states Ψ_A, Ψ_B by $\cos^2 c = |\langle A, B\rangle|^2$. The quantity $|\langle A, B\rangle|^2$ is called *interference* between the rays $[A], [B]$, and c is sometimes called the 'Bargmann angle' between the state vectors $|A\rangle, |B\rangle$ [18]; this makes sense in $\mathcal{H}$ but on $\mathcal{P}$ 'distance' seems a more appropriate name, as c measures the *separation* between states. For fixed $|A\rangle, |B\rangle$, there are two possible values, c and $\pi - c$, determined by the former expression; both are distances along the 'short' and 'long' geodesic arcs joining $[A]$ to $[B]$ which taken together complete a closed geodesic (the FS distance between two orthogonal states is $\pi/2$; this is the maximal *unoriented* distance between any two states).

Let us now recall the known $\mathcal{P}$ invariants of three states Ψ_A, Ψ_B, Ψ_C, for which $|A\rangle, |B\rangle, |C\rangle$ represent vectors in $\mathcal{H}$. In addition to $|\langle A, B\rangle|, |\langle B, C\rangle|, |\langle C, A\rangle|$, the cyclic product

$$\langle A, B, C\rangle := \langle A, B\rangle\langle B, C\rangle\langle C, A\rangle \tag{2.1}$$

is a geometric invariant in $\mathcal{P}$, since the eventual phase freedom in each vector state $|A\rangle, |B\rangle, |C\rangle$ cancels out and $\langle A, B, C\rangle$ depends only on the rays $[A], [B], [C]$. This three-state invariant, which is a *complex* number, was introduced in physics by Bargmann (as a tool for distinguishing unitary and antiunitary transformations in his celebrated proof of Wigner's theorem) [8] but was geometrically considered much earlier in its proper trigonometric context by Blaschke and Terheggen [7]; therefore it will be called here the Blaschke–Terheggen–Bargmann (BTB) three-state (triangle or triangular) invariant.

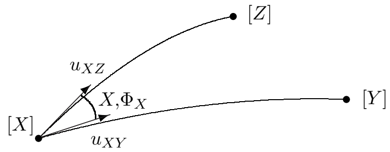

Figure 1. Hermitian angle and angular phase at the vertex $[X]$.

Since the modulus $|\langle A, B, C\rangle| = |\langle A, B\rangle||\langle B, C\rangle||\langle C, A\rangle|$ is the product of the three two-state invariants, the essential new content of $\langle A, B, C\rangle$ is embodied in a phase Ω, which as we will see in (4.2) is related to the symplectic area of the triangle $\widehat{ABC}$:

$$e^{-i\Omega} = \frac{\langle A, B\rangle\langle B, C\rangle\langle C, A\rangle}{|\langle A, B\rangle| \cdot |\langle B, C\rangle| \cdot |\langle C, A\rangle|}. \tag{2.2}$$

The invariants just discussed do not exhaust the natural ones associated with the three points in $\mathcal{P}$. To discuss this it is better to devote some time to the analysis of invariants at a vertex formed by two geodesics meeting at a point in $\mathcal{P}$, say $[X]$. There are *two* independent angular invariants, associated with the two commuting factors in $U(1) \otimes SU(2)$. A choice for them was made on group theoretical grounds in [3]. Here we provide a more conventional alternative definition of these invariants in terms of Hermitian products between the tangent vectors to the FS geodesics at $[X]$. First of all let us give an explicit expression for u_{XY}, the tangent vector at $[X]$ to the (short) geodesic arc joining two (non-orthogonal) points $[X]$, $[Y]$ in $\mathcal{P}$. The tangent space $T_{[X]}$ at an arbitrary point $[X]$ in $\mathcal{P}$ can be identified with the subspace $[X]^\perp \subset \mathbb{C}^3$ which is Hermitian orthogonal to the vector $|X\rangle$, so that for any tangent vector u at $[X]$ we have $\langle X, u\rangle = 0$. As the action of $SU(3)$ in $\mathcal{H} \equiv \mathbb{C}^3$ is linear, the vector u_{XY} should be a linear combination of $|X\rangle$ and $|Y\rangle$. The Hermitian orthogonality condition makes u_{XY} proportional to $|Y\rangle - \langle X, Y\rangle|X\rangle$. Now, as $\langle X, Y\rangle \neq 0$, we may add a conventional factor $\langle Y, X\rangle$ in order to make

$$u_{XY} := \langle Y, X\rangle(|Y\rangle - \langle X, Y\rangle|X\rangle) \tag{2.3}$$

insensitive to the phase freedom in the vector representing the endpoint $[Y]$. The vector u_{XY} still changes by a phase factor if the vector representing the vertex $[X]$ does; in this sense this only gives a *non-canonical* representation of $T_{[X]}$ at the point $[X]$ in the subspace $[X]^\perp \subset \mathbb{C}^3$. But for any Hermitian product $\langle u_{XY}, u_{XZ}\rangle$, with $[Z]$ any third point in $\mathcal{P}$, this phase ambiguity is irrelevant, as long as the *same* representing vector for $|X\rangle$ is used in both tangent vectors.

We can now *define* the two real $\mathcal{P}$-invariants, the *Hermitian angle X* and the *angular phase* Φ_X, which collectively describe the 'complete (complex) angle' between u_{XY} and u_{XZ} at the vertex where the two (short) geodesics linking $[X]$ to $[Y]$ and to $[Z]$ meet (figure 1):

$$\cos X\, e^{i\Phi_X} := \frac{\langle u_{XY}, u_{XZ}\rangle}{\|u_{XY}\| \cdot \|u_{XZ}\|} = \frac{\overline{\langle X, Y\rangle\langle Y, Z\rangle\langle Z, X\rangle} - |\langle X, Y\rangle|^2|\langle X, Z\rangle|^2}{|\langle X, Y\rangle| \cdot |\langle X, Z\rangle|\sqrt{1 - |\langle X, Y\rangle|^2}\sqrt{1 - |\langle X, Z\rangle|^2}}. \tag{2.4}$$

The first equality is the definition for (X, Φ_X) and the second follows after replacing the tangent vectors u_{XY}, u_{XZ} by means of (2.3). The angular phase Φ_X is *undefined* when $|Y\rangle$ or $|Z\rangle$ is orthogonal to $|X\rangle$; the trivial two-valuedness $(\pi - X, \Phi_X) \equiv (X, \Phi_X + \pi)$ may be resolved by reducing the range of X to the interval $[0, \pi/2)$ and leaving Φ_X in the full range $[0, 2\pi)$ (compare [19]). Although the tangent vectors to $\mathcal{P}$ are identified through (2.3) with vectors u_{XY}, u_{XZ} in the ambient space $\mathbb{C}^3$ once the representation is fixed, these vectors have *no phase freedom* at all, and u_{XY}, $e^{i\epsilon}u_{XY}$ are tangent to two *different* FS geodesics; hence Φ_X is actually an invariant in $\mathcal{P}$. A potentially confusing issue must be cleared: the 'angular

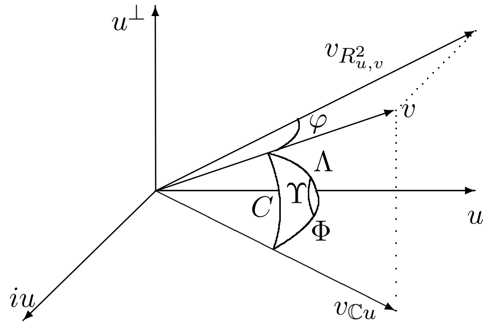

Figure 2. Angular invariants of a vertex formed by two tangent vectors u, v at a point in $\mathcal{P}$, interpreted as 'ordinary' Fubini–Study angles.

phase' between two position vectors $|X\rangle, |Y\rangle$ in $\mathcal{H}$ is meaningless as a quantity in $\mathcal{P}$, yet the angular phases between two geodesics in $\mathcal{P}$ are meaningful.

Once the angle at a Hermitian vertex is recognized as a 'two-component' quantity, it is clear that there are many possible choices for vertex invariants (they are reviewed in [19]). Two further natural invariants Λ_X and φ_X are defined as

$$\Lambda_X := \arccos\left(\frac{\mathrm{Re}\langle u_{XY}, u_{XZ}\rangle}{\|u_{XY}\|\cdot\|u_{XZ}\|}\right) \qquad \varphi_X := \arcsin\left(\frac{\mathrm{Im}\langle u_{XY}, u_{XZ}\rangle}{\|u_{XY}\|\cdot\|u_{XZ}\|}\right). \tag{2.5}$$

Λ_X, sometimes called the *Euclidean angle* between u_{XY}, u_{XZ}, can be directly interpreted, from its definition (2.5), as the 'ordinary' angle between these vectors in the Riemannian FS structure, and thus we propose to call it the *FS angle*. It is possible to find an interpretation for the remaining angular invariants as 'ordinary' FS angles between appropriately chosen tangent vectors. Let u, v be any two tangent vectors to $\mathcal{P}$ at $[X]$, and let us call $\mathbb{C}_u$ the subspace of the tangent space $T_{[X]}$ at $[X]$ generated by u, iu. The Hermitian projection of v onto $\mathbb{C}_u$ is a tangent vector $v_{\mathbb{C}u}$ at $[X]$:

$$v_{\mathbb{C}u} := \frac{\langle u, v\rangle}{\|u\|^2}u = \left(\frac{\mathrm{Re}\langle u, v\rangle}{\|u\|^2} + \mathrm{i}\frac{\mathrm{Im}\langle u, v\rangle}{\|u\|^2}\right)u = \frac{\mathrm{Re}\langle u, v\rangle}{\|u\|^2}u + \frac{\mathrm{Re}\langle \mathrm{i}u, v\rangle}{\|u\|^2}(\mathrm{i}u). \tag{2.6}$$

Let us call $\mathbb{R}^2_{u,v}$ the subspace of $T_{[X]}$ generated by u and $u^\perp := v - v_{\mathbb{C}u}$. The vector sum of the FS 'orthogonal' (not the Hermitian) projections of v onto u and $u^\perp$ is a tangent vector $v_{R^2_{u,v}}$ at $[X]$:

$$v_{R^2_{u,v}} := \frac{\mathrm{Re}\langle u, v\rangle}{\|u\|^2}u + \frac{\mathrm{Re}\langle u^\perp, v\rangle}{\|u^\perp\|^2}u^\perp. \tag{2.7}$$

All the tangent vectors u, v, $v_{\mathbb{C}u}$ and $v_{R^2_{u,v}}$ to $\mathcal{P}$ at $[X]$ live in a three-dimensional subspace of $T_{[X]}$, schematically displayed in figure 2. It is possible to write the four angular invariants $X, \Lambda, \varphi, \Phi$ of a pair of tangent vectors u, v as angles of the Riemannian FS structure between pairs of vectors taken from u, v, $v_{\mathbb{C}u}$ and $v_{R^2_{u,v}}$:

$$\begin{aligned} \cos\Lambda &= \frac{\mathrm{Re}\langle u, v\rangle}{\|u\|\cdot\|v\|} \qquad & \cos\varphi &= \frac{\mathrm{Re}\left\langle v, v_{R^2_{u,v}}\right\rangle}{\|v\|\cdot\left\|v_{R^2_{u,v}}\right\|} \\ \cos X &= \frac{\mathrm{Re}\langle v, v_{\mathbb{C}u}\rangle}{\|v\|\cdot\|v_{\mathbb{C}u}\|} \qquad & \cos\Phi &= \frac{\mathrm{Re}\langle u, v_{\mathbb{C}u}\rangle}{\|u\|\cdot\|v_{\mathbb{C}u}\|}. \end{aligned} \tag{2.8}$$

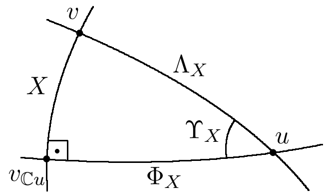

Figure 3. 'Verticilar triangle' formed by several angular invariants associated with a pair u, v of tangent vectors in $\mathcal{P}$; these, as well as $v_{\mathbb{C}u}$, are the vertices of the verticilar triangle.

There is another important invariant: the *holomorphy inclination* Υ (holomorphy angle, Kähler angle) which is actually defined for any real 2-flat spanned by u, v. It is an invariant of the real tangent 2-flat and therefore is more an 'inclination of span(u, v)' than an 'angle between u, v'. It can also be expressed in terms of the FS Riemannian structure: when two vectors u, w are chosen so that they span the same (real) 2-flat as u, v and are further FS orthogonal, the holomorphy inclination of the 2-flat is

$$\Upsilon = \arccos\left(\frac{\mathrm{Re}\langle \mathrm{i}u, w\rangle}{\|u\| \cdot \|w\|}\right) \qquad \text{when} \quad \mathrm{Re}\langle u, w\rangle = 0. \tag{2.9}$$

Since Υ is the FS angle between iu and the vector w spanning together with u the same 2-flat as u, v but FS orthogonal to u, we can interpret the holomorphy inclination of the given 2-flat as measuring how it separates from the real 2-flat $\mathbb{C}_u$ (or $\mathbb{C}_v$) containing u (or v) and *invariant* under the complex structure (see figure 2). The holomorphy inclination of a submanifold of $\mathbb{C}P^1$ type at each point is zero, as these submanifolds are invariant under the complex structure in $\mathcal{P}$; for the submanifolds of $\mathbb{R}P^2$ type its holomorphy inclination at each point equals $\pi/2$. The relation of Υ with the other angular invariants we have considered is depicted in figure 3, which displays the 'verticilar triangle' associated with the vertex $[X]$, where the two FS geodesics in $\mathcal{P}$ with tangent vectors u, v meet.

The important point here is that *two independent invariants* are required to *completely* describe the relative position of two intersecting FS geodesics; hence it is better to think of the angle in $\mathcal{P}$ as an intrinsically *two-component quantity*. In [3] it is shown that (X, Φ_X) are linked to a pair of commuting isometries of $\mathcal{P}$ and in this sense they are the most natural choice to obtain a complete set of self-dual trigonometric equations; these are related [19] to other vertex invariants by

$$\begin{array}{ll}
\cos\Lambda_X = \cos X \cos\Phi_X & \cos\Upsilon_X \sin\Lambda_X = \cos X \sin\Phi_X \\
\sin X = \sin\Lambda_X \sin\Upsilon_X & \sin\varphi_X = \sin\Lambda_X \cos\Upsilon_X \\
\cos^2 X = \cos^2\Lambda_X + \sin^2\Lambda_X \cos^2\Upsilon_X & \sin^2\Lambda_X = \sin^2 X + \sin^2\varphi_X \\
\tan\Phi_X = \cos\Upsilon_X \tan\Lambda_X & \tan X = \tan\Upsilon_X \sin\Phi_X .
\end{array} \tag{2.10}$$

Let us now return to triangles. Consider the FS geodesics in $\mathcal{P}$ joining the pairs out of three points $[A], [B], [C]$, which are generically unique (except if their endpoint states are separated by $\pi/2$ in $\mathcal{P}$), and let us choose for sides g_a, g_b, g_c the short arcs on the geodesics joining every two vertices. At each vertex, we can define two angular invariants A, Φ_A; B, Φ_B; C, Φ_C as in (2.4); remark that at vertex $[A]$ we are choosing the *external angle* (see figure 4) for reasons explained in [2, 3].

The choice of capital letters for angular invariants allows a clear and systematic typographic rendering (uppercase/lowercase change) of the self-duality of the equations

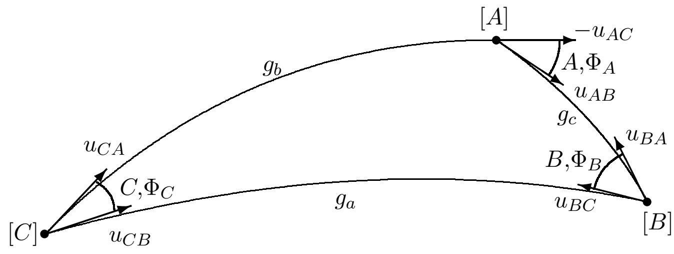

Figure 4. Angular invariants at the three triangle vertices.

we will propose. Taken together, the definitions for the angular invariants at the three vertices can be considered as trigonometric equations of $\mathcal{P}$ and imply that all triangular invariants a, b, c; A, B, C; Φ_A, Φ_B, Φ_C; Ω introduced so far can be expressed in terms of the three side lengths a, b, c and the BTB phase Ω (i.e. the three two-vertex invariants $|\langle A, B\rangle|, |\langle B, C\rangle|, |\langle C, A\rangle|$ and a three-vertex invariant $\langle A, B, C\rangle$); therefore a triangle in $\mathcal{P}$ (or in $\mathbb{C}P^2$) is determined up to isometries by four real quantities [11].

This approach to trigonometry in $\mathcal{P}$ has two different drawbacks. A minor one lies in the restrictions $a, b, c < \pi/2$ to the short geodesic arc for the triangle sides; the approach through Hermitian products becomes indeterminate when a side equals $\pi/2$ as in this case the geodesic joining the two vertices is not unique, and thus is not determined by the vertices alone. More serious is the loss of manifest self-duality: while Hermitian elliptic space *is self-dual* by the usual polarity relation (exactly similar to the real elliptic space or its covering, the sphere), the self-duality is *not explicitly present* in this formulation of trigonometry of $\mathcal{P}$ because of the asymmetric role sides and angles play.

Both disadvantages are avoided at once in the approach proposed in [3]. The imprecise idea of 'triangle as three points' is replaced by the concept of *triangular loop* whose data are three points *plus* three geodesic arcs (length $\pi/2$ allowed) joining them, and invariants are defined as canonical parameters of suitably chosen one-parameter subgroups. This leads to the explicit introduction of yet another *lateral phase* by dualizing the definition of angular invariants, thus restoring the side/vertex duality. As far as we know this has not been performed before, yet it makes everything stand out clearly. Each side, say g_a, has a well-defined *pole*, represented by a vector $|\psi_a\rangle \equiv |a\rangle$, for which $[a]$ is at a distance $\pi/2$ from *all* points in g_a, thus $|a\rangle$ should satisfy $\langle a, B\rangle = \langle a, C\rangle = 0$. From elementary linear algebra, if the vectors $|B\rangle, |C\rangle$ are given in terms of an orthonormal basis of $\mathbb{C}^3$ as $|B\rangle = \sum B_i|i\rangle$, $|C\rangle = \sum C_i|i\rangle$, then the bra position vector $\langle\psi_a| \equiv \langle a|$ of the pole of the side a is proportional to the 'bra vector product' $\langle\overline{B \times C}| := \sum\langle i|\epsilon_{ijk}B_jC_k$, which obviously warrants $\langle\overline{B \times C}, B\rangle = 0$, $\langle\overline{B \times C}, C\rangle = 0$, as required.

The three ket vectors $|a\rangle, |b\rangle, |c\rangle$ associated with the bras $\langle\psi_a|, \langle\psi_b|, \langle\psi_c| \equiv \langle a|, \langle b|, \langle c|$ determine a triangle $\widehat{abc}$ (figure 5) 'polar' to the initial one $\widehat{ABC}$, and all the invariants already given for the triangle $\widehat{ABC}$ can now be defined for the polar triangle; these definitions ensure that the angular invariants of the triangle $\widehat{abc}$ (associated with its vertices) are the searched lateral invariants of the initial triangle $\widehat{ABC}$; indeed these coincide with the invariants defined group theoretically in [3].

As these two triangles are 'polar' to each other, we could start from the triangle $\widehat{abc}$ and (re-)define the vectors $|A\rangle, |B\rangle, |C\rangle$ in terms of $|a\rangle, |b\rangle, |c\rangle$. The (normalized) *'polar' vector states* $|a\rangle, |b\rangle, |c\rangle$ in terms of the original vector states $|A\rangle, |B\rangle, |C\rangle$, and the vectors $|A'\rangle, |B'\rangle, |C'\rangle$ obtained in the same way from $|a\rangle, |b\rangle, |c\rangle$ are defined as

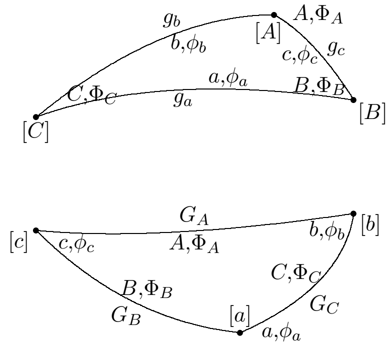

Figure 5. A triangle $\widehat{ABC}$ in $\mathcal{P}$ and its dual triangle $\widehat{abc}$. The vertices of the original triangle are the poles of the sides of the polar triangle and conversely.

$$
\begin{aligned}
\langle a| &= \frac{-1}{\sqrt{1-|\langle B, C\rangle|^2}} \langle \overline{B \times C}| & \langle A'| &= \frac{-1}{\sqrt{1-|\langle b, c\rangle|^2}} \langle \overline{b \times c}| \\
\langle b| &= \frac{1}{\sqrt{1-|\langle C, A\rangle|^2}} \langle \overline{C \times A}| & \langle B'| &= \frac{1}{\sqrt{1-|\langle c, a\rangle|^2}} \langle \overline{c \times a}| \\
\langle c| &= \frac{1}{\sqrt{1-|\langle A, B\rangle|^2}} \langle \overline{A \times B}| & \langle C'| &= \frac{1}{\sqrt{1-|\langle a, b\rangle|^2}} \langle \overline{a \times b}|.
\end{aligned}
\tag{2.11}
$$

Note the sign in the definition of $\langle A|$ and $\langle a|$; this is related to the consideration of the 'external angles' at $[A]$ and will appear consistently in all expressions. Of course $|A'\rangle, |B'\rangle, |C'\rangle$ and $|A\rangle, |B\rangle, |C\rangle$ are proportional; the exact relations between them are

$$
|A'\rangle = \frac{\langle a, A\rangle}{\sin b \sin C} |A\rangle \qquad |B'\rangle = \frac{-\langle b, B\rangle}{\sin c \sin A} |B\rangle \qquad |C'\rangle = \frac{-\langle c, C\rangle}{\sin a \sin B} |C\rangle. \tag{2.12}
$$

By taking into account $\langle a, A\rangle \sin a = -\langle b, B\rangle \sin b = -\langle c, C\rangle \sin c$ (similar relations hold with $\sin A, \sin B, \sin C$ instead of $\sin a, \sin b, \sin c$), we can write relations for the Hermitian products of these vector states:

$$
\begin{aligned}
\langle A, B\rangle &= \frac{\langle a, b\rangle - \langle a, c\rangle\langle c, b\rangle}{\sqrt{1-|\langle c, a\rangle|^2}\sqrt{1-|\langle b, c\rangle|^2}} & \langle a, b\rangle &= \frac{\langle A, B\rangle - \langle A, C\rangle\langle C, B\rangle}{\sqrt{1-|\langle C, A\rangle|^2}\sqrt{1-|\langle B, C\rangle|^2}} \\
\langle B, C\rangle &= \frac{-\langle b, c\rangle + \langle b, a\rangle\langle a, c\rangle}{\sqrt{1-|\langle a, b\rangle|^2}\sqrt{1-|\langle c, a\rangle|^2}} & \langle b, c\rangle &= \frac{-\langle B, C\rangle + \langle B, A\rangle\langle A, C\rangle}{\sqrt{1-|\langle A, B\rangle|^2}\sqrt{1-|\langle C, A\rangle|^2}} \\
\langle C, A\rangle &= \frac{\langle c, a\rangle - \langle c, b\rangle\langle b, a\rangle}{\sqrt{1-|\langle a, b\rangle|^2}\sqrt{1-|\langle b, c\rangle|^2}} & \langle c, a\rangle &= \frac{\langle C, A\rangle - \langle C, B\rangle\langle B, A\rangle}{\sqrt{1-|\langle A, B\rangle|^2}\sqrt{1-|\langle B, C\rangle|^2}}.
\end{aligned}
\tag{2.13}
$$

The obvious triangular invariants are the sides of these two triangles, defined through the modulus of Hermitian products of the corresponding vector states, $|A\rangle, |B\rangle, |C\rangle, |a\rangle, |b\rangle, |c\rangle$ (which are all considered to be of modulus 1):

$$
\begin{aligned}
\cos A &= |\langle b, c\rangle| & \cos a &= |\langle B, C\rangle| \\
\cos B &= |\langle c, a\rangle| & \cos b &= |\langle C, A\rangle| \\
\cos C &= |\langle a, b\rangle| & \cos c &= |\langle A, B\rangle|.
\end{aligned}
\tag{2.14}
$$

But we have also defined 'complete angles' (Hermitian angle C and also a companion *angular phase*, Φ_C) at $[C]$ through the Hermitian product of two tangent vectors. The tangent vectors to the two (short) geodesics which join vertices to each other are

$$\begin{aligned}
u_{AB} &= \langle B, A\rangle(|B\rangle - \langle A, B\rangle|A\rangle) & u_{ab} &= \langle b, a\rangle(|b\rangle - \langle a, b\rangle|a\rangle)\\
u_{AC} &= \langle C, A\rangle(|C\rangle - \langle A, C\rangle|A\rangle) & u_{ac} &= \langle c, a\rangle(|c\rangle - \langle a, c\rangle|a\rangle)\\
u_{BA} &= \langle A, B\rangle(|A\rangle - \langle B, A\rangle|B\rangle) & u_{ba} &= \langle a, b\rangle(|a\rangle - \langle b, a\rangle|b\rangle)\\
u_{BC} &= \langle C, B\rangle(|C\rangle - \langle B, C\rangle|B\rangle) & u_{bc} &= \langle c, b\rangle(|c\rangle - \langle b, c\rangle|b\rangle)\\
u_{CA} &= \langle A, C\rangle(|A\rangle - \langle C, A\rangle|C\rangle) & u_{ca} &= \langle a, c\rangle(|a\rangle - \langle c, a\rangle|c\rangle)\\
u_{CB} &= \langle B, C\rangle(|B\rangle - \langle C, B\rangle|C\rangle) & u_{cb} &= \langle b, c\rangle(|b\rangle - \langle c, b\rangle|c\rangle)
\end{aligned} \tag{2.15}$$

and from them, by using expressions similar to (2.4), we obtain definitions for the 'complete angles' at the three vertices *and dual definitions for the 'complete sides' (distances* a, b, c and lateral phases ϕ_a, ϕ_b, ϕ_c) (recall that the angles at $[A]$ are the external ones):

$$\begin{aligned}
\cos A\, \mathrm{e}^{-\mathrm{i}\Phi_A} &= \frac{\langle -u_{AC}, u_{AB}\rangle}{\|u_{AC}\|\cdot\|u_{AB}\|} & \cos a\, \mathrm{e}^{-\mathrm{i}\phi_a} &= \frac{\overline{\langle -u_{ac}, u_{ab}\rangle}}{\|u_{ac}\|\cdot\|u_{ab}\|}\\
\cos B\, \mathrm{e}^{\mathrm{i}\Phi_B} &= \frac{\langle u_{BA}, u_{BC}\rangle}{\|u_{BA}\|\cdot\|u_{BC}\|} & \cos b\, \mathrm{e}^{\mathrm{i}\phi_b} &= \frac{\overline{\langle u_{ba}, u_{bc}\rangle}}{\|u_{ba}\|\cdot\|u_{bc}\|}\\
\cos C\, \mathrm{e}^{\mathrm{i}\Phi_C} &= \frac{\langle u_{CB}, u_{CA}\rangle}{\|u_{CB}\|\cdot\|u_{CA}\|} & \cos c\, \mathrm{e}^{\mathrm{i}\phi_c} &= \frac{\overline{\langle u_{cb}, u_{ca}\rangle}}{\|u_{cb}\|\cdot\|u_{ca}\|}.
\end{aligned} \tag{2.16}$$

As far as sides a, b, c and Hermitian angles A, B, C alone are concerned, (2.16) implies (2.14) (use (2.13)). But the *new* content of equations (2.16) lies in the companion definition they provide for the three angular phases of the triangle $\widehat{ABC}$ and for the three new *lateral phases* associated with the sides of the initial triangle (defined as angular phases of the polar triangle $\widehat{abc}$). To finish, a triangular invariant polar or dual to the BTB triangular invariant Ω can also be defined; it will be denoted by ω. The expressions for these two invariants Ω, ω in terms of Hermitian products of vector states and 'polar' vector states, together with similar expressions for the angular phases Φ_A, Φ_B, Φ_C and lateral phases ϕ_a, ϕ_b, ϕ_c, are easily found from (2.14) and (2.16):

$$\begin{aligned}
\mathrm{e}^{\mathrm{i}\Omega} &= \frac{\overline{\langle A, B\rangle\langle B, C\rangle\langle C, A\rangle}}{|\langle A, B\rangle|\cdot|\langle B, C\rangle|\cdot|\langle C, A\rangle|} & \mathrm{e}^{\mathrm{i}\omega} &= \frac{\langle a, b\rangle\langle b, c\rangle\langle c, a\rangle}{|\langle a, b\rangle|\cdot|\langle b, c\rangle|\cdot|\langle c, a\rangle|}\\
\mathrm{e}^{-\mathrm{i}\Phi_A} &= \frac{\overline{\langle A, B\rangle\langle b, c\rangle\langle C, A\rangle}}{|\langle A, B\rangle|\cdot|\langle b, c\rangle|\cdot|\langle C, A\rangle|} & \mathrm{e}^{-\mathrm{i}\phi_a} &= \frac{\langle a, b\rangle\langle B, C\rangle\langle c, a\rangle}{|\langle a, b\rangle|\cdot|\langle B, C\rangle|\cdot|\langle c, a\rangle|}\\
\mathrm{e}^{\mathrm{i}\Phi_B} &= \frac{\overline{\langle A, B\rangle\langle B, C\rangle\langle c, a\rangle}}{|\langle A, B\rangle|\cdot|\langle B, C\rangle|\cdot|\langle c, a\rangle|} & \mathrm{e}^{\mathrm{i}\phi_b} &= \frac{\langle a, b\rangle\langle b, c\rangle\langle C, A\rangle}{|\langle a, b\rangle|\cdot|\langle b, c\rangle|\cdot|\langle C, A\rangle|}\\
\mathrm{e}^{\mathrm{i}\Phi_C} &= \frac{\overline{\langle a, b\rangle\langle B, C\rangle\langle C, A\rangle}}{|\langle a, b\rangle|\cdot|\langle B, C\rangle|\cdot|\langle C, A\rangle|} & \mathrm{e}^{\mathrm{i}\phi_c} &= \frac{\langle A, B\rangle\langle b, c\rangle\langle c, a\rangle}{|\langle A, B\rangle|\cdot|\langle b, c\rangle|\cdot|\langle c, a\rangle|}.
\end{aligned} \tag{2.17}$$

The Bargmann relation (2.2) thus appears as a particular case of a kind of relation of similar structure expressing all *phase invariants* (three lateral phases, three angular phases and two BTB invariants) in terms of vector states and dual vector states.

3. The trigonometry in the quantum space of states

Equations (2.17) embody the basic relations among the trigonometric invariants introduced so far. To get rid of objects in the Hilbert vector space (such as the position vectors

$|C\rangle, |c\rangle \ldots$ or the vectors $u_{CA}, u_{ca}, \ldots$ representing the tangent vectors) it suffices to replace $\langle a, b\rangle, \langle b, c\rangle, \langle c, a\rangle$ from (2.13) in (2.17) and then use (2.14):

$$\begin{aligned} &-\phi_a + \Phi_B + \Phi_C = -\Phi_A + \phi_b + \Phi_C = -\Phi_A + \Phi_B + \phi_c = \Omega \\ &-\Phi_A + \phi_b + \phi_c = -\phi_a + \Phi_B + \phi_c = -\phi_a + \phi_b + \Phi_C = \omega. \end{aligned} \tag{3.1}$$

$$\begin{aligned} \cos A\, e^{-i\Phi_A} &= \frac{\cos a\, e^{i\Omega} - \cos b \cos c}{-\sin b \sin c} & \cos a\, e^{-i\phi_a} &= \frac{\cos A\, e^{i\omega} - \cos B \cos C}{-\sin B \sin C} \\ \cos B\, e^{i\Phi_B} &= \frac{\cos b\, e^{i\Omega} - \cos c \cos a}{\sin c \sin a} & \cos b\, e^{i\phi_b} &= \frac{\cos B\, e^{i\omega} - \cos C \cos A}{\sin C \sin A} \\ \cos C\, e^{i\Phi_C} &= \frac{\cos c\, e^{i\Omega} - \cos a \cos b}{\sin a \sin b} & \cos c\, e^{i\phi_c} &= \frac{\cos C\, e^{i\omega} - \cos A \cos B}{\sin A \sin B}. \end{aligned} \tag{3.2}$$

These equations completely contain the trigonometry of the quantum space of states. All of them *explicitly display self-duality*, and not all of them can be independent. Both (3.1) and (3.2) also follow from the method developed in [3] which is not hindered by the technical restrictions $a, b, c < \pi/2$ (or $A, B, C < \pi/2$); thus these equations turn out to hold irrespective of these restrictions which however we had to enforce in the derivation presented here. In the group-theoretical approach [3], only $a, b, c, \phi_a, \phi_b, \phi_c$ and $A, B, C; \Phi_A, \Phi_B, \Phi_C$ are the 'primary invariants' and neither Ω, ω nor equations (3.1) appear at the outset, but rather *follow* from other trigonometric relations,

$$-\Phi_A + \phi_a = \Phi_B - \phi_b = \Phi_C - \phi_c \tag{3.3}$$

which involve only angular and lateral phases, irrespective of the values of the sides and Hermitian angles. These are related to the Cartan subalgebra of the rank-two algebra $\mathfrak{su}(3)$, and have no siblings in the trigonometry of the real sphere S^2 associated with the rank-one algebra $\mathfrak{so}(3)$. As there are two independent relations between the six phases, only four phases could be independent. Thus the equalities

$$\begin{aligned} &-\phi_a + \Phi_B + \Phi_C = -\Phi_A + \phi_b + \Phi_C = -\Phi_A + \Phi_B + \phi_c \\ &-\Phi_A + \phi_b + \phi_c = -\phi_a + \Phi_B + \phi_c = -\phi_a + \phi_b + \Phi_C \end{aligned} \tag{3.4}$$

follow from (3.3) and the common values for these two quantities are *defined* within this alternative group theoretical approach as two triangle invariants Ω, ω. Whatever way (3.1) is obtained, equations (3.1) and (3.2) are the basic trigonometric equations. The *Cartan sector* equations (3.1) and (3.3) are extremely important, as they relate the two triangle phases Ω, ω to 'mixed' phase excesses and imply that the common value in (3.3) also equals $\Omega - \omega$. The remaining basic equations (3.2) are called the (complex) Hermitian cosine theorem for sides and the dual Hermitian cosine theorem for angles.

Once (3.1) is assumed, the two sets in (3.2) are equivalent, and thus contain six independent real equations. Taken together there are ten independent equations relating the 14 invariants; thus a triangle in $\mathcal{P}$ is determined, up to an $SU(3)$ motion, by four real invariants. A rather large number of simple equations can be derived from these basic ones, and are shown in the appendix; they are given in [3] in the 'general CKD form'.

The triangle symplectic area $\mathcal{S}$, defined as the integral of the symplectic 2-form in $\mathcal{P}$ over any surface bounded by the geodesic triangle, turns out [21] to be proportional to the BTB invariant Ω, with our normalization conventions

$$2\mathcal{S} = \Omega \tag{3.5}$$

so that the symplectic area is proportional to the *mixed* (two *angular* and one *lateral*) *phase excess*:

$$2\mathcal{S} = \Omega = -\phi_a + \Phi_B + \Phi_C = -\Phi_A + \phi_b + \Phi_C = -\Phi_A + \Phi_B + \phi_c. \tag{3.6}$$

Recall the name excess used here fits with the usual meaning; had internal angles been used at A, then all these excesses would be the *sum of three phases minus* π (see [2] and comments in section 4.6 on angular excesses). For other normalization choices we would have $(K_{\rm hol}/2)\mathcal{S} = \Omega$; here the constant holomorphic curvature of $\mathcal{P}$ is $K_{\rm hol} = 4$. When $K_{\rm hol} \to 0$, then Ω vanishes, but the symplectic area keeps some finite value, independent of the angular and lateral phases, just as in the situation for area and angular excess in the real 'spherical versus Euclidean' trigonometry. As everything in the approach is self-dual, completely similar results hold, mutatis mutandis, for the dual quantities s and ω:

$$2s = \omega = -\Phi_A + \phi_b + \phi_c = -\phi_a + \Phi_B + \phi_c = -\phi_a + \phi_b + \Phi_C. \tag{3.7}$$

The two quantities Ω, ω can be replaced in all trigonometric equations by symplectic area and its dual quantity, through (3.6) and (3.7). In particular, the Euler-like equations (A.34) and (A.35) can be rewritten as

$$\sin(2\mathcal{S}) = \frac{\sin\Phi_A \sin\Phi_B \sin^2 c}{\sin\phi_c} \qquad \sin(2s) = \frac{\sin\phi_a \sin\phi_b \sin^2 C}{\sin\Phi_C}. \tag{3.8}$$

Note that together with the 'mixed' phase excesses ω, Ω, 'pure angular' and 'pure lateral' phase excesses Δ_Φ, δ_ϕ can be defined; these are related to Ω, ω by

$$\Delta_\Phi := -\Phi_A + \Phi_B + \Phi_C = 2\Omega - \omega \qquad \delta_\phi := -\phi_a + \phi_b + \phi_c = 2\omega - \Omega. \tag{3.9}$$

4. Physical interpretation of trigonometry in $\mathcal{P}$

Once trigonometry for a triangle in $\mathcal{P}$ has been completely established, let us turn to its physical interpretation. A point in $\mathcal{P}$ represents a state in which a quantum system can be found, and a curve in $\mathcal{P}$ represents its evolution, which can be either a continuous evolution according to the Schödinger equation or a discontinuous change when a filtering measurement is performed; these can be described by FS geodesics in $\mathcal{P}$ and are the type we will consider.

Any three vectors representing the states Ψ_A, Ψ_B, Ψ_C lie generically on a $\mathbb{C}^3$ subspace of $\mathcal{H}$ (spanned either by Ψ_A, Ψ_B, Ψ_C or by ψ_a, Ψ_B, Ψ_C) and all the discussion will be done in this subspace, whose associated quantum state space is a $\mathbb{C}P^2$. For the triangle determined by these states and short geodesic arcs joining them, the three sides and symplectic area are *independent*, although they satisfy the inequality [3]

$$2\cos a\cos b\cos c\cos(2\mathcal{S}) \geqslant \cos^2 a + \cos^2 b + \cos^2 c - 1. \tag{4.1}$$

4.1. Bargmann interferences and distances

The three sides a, b, c have an immediate meaning, which is indeed a two-point one: for any two states, say Ψ_A, Ψ_B, $\cos^2 c$ is the probability of finding, upon a complete measurement, the state Ψ_B when the state Ψ_A is known. Thus these invariants convey the same content as the Bargmann interference between states. Of course, there is a similar interpretation for a, b. If Γ is any other curve joining Ψ_A and Ψ_B (say the image of some time evolution according to the Schrödinger equation), the FS distance along Γ is, physically, the integral over time of the *dispersion* of the Hamiltonian [20].

4.2. Geometric phases, symplectic area and the BTB invariant

In the conventional Hilbert space language, if a filtering measurement is actually performed on a quantum system that is known to be in the state Ψ_A, and is found in Ψ_B, the new vector state after such a measurement is related to the state before by

$$|A\rangle \to \langle B, A\rangle |B\rangle \tag{4.2}$$

so that the 'projected' state vector is 'in phase' with the initial one. This process is ultimately responsible for the geometrical phase that a state acquires after a cyclic evolution. Three such filtering measurements are required to make a quantum system undergo the simplest evolution that takes it back to its original state, and after this evolution the phase Σ 'accumulated' by the system happens to be exactly the BTB phase invariant:

$$\Psi_A \to \Psi_B \to \Psi_C \to \Psi_A \qquad |A\rangle \to \overline{\langle A, B, C\rangle}|A\rangle = \mathrm{e}^{\mathrm{i}\Sigma}|A\rangle. \tag{4.3}$$

Thus there are relations linking the quantities $\Sigma, \mathcal{S}$ and $\langle A, B, C\rangle$ to trigonometric invariants of the triangle $\Omega, \Phi_A, \phi_a, \ldots$. For any loop in $\mathcal{P}$, Anandan [22] proved that twice the symplectic area $2\mathcal{S}$ (with our normalization) equals the geometric phase Σ associated with the cyclic evolution along the loop. Bringing this together with (3.5), (3.6) and (4.3), we may write for any triangle loop

$$\Sigma = 2\mathcal{S} = \Omega = -\phi_a + \Phi_B + \Phi_C = -\Phi_A + \phi_b + \Phi_C = -\Phi_A + \Phi_B + \phi_c = -\arg\langle A, B, C\rangle. \tag{4.4}$$

The relation $2\mathcal{S} = -\arg\langle A, B, C\rangle$ [23] has also been recently discussed [24]; we claim that the proper context of this relation is the trigonometry of a triangular loop in $\mathcal{P}$. In any case, the triangular BTB phase and the geometric phase are quantities belonging to the intrinsic geometry of $\mathcal{P}$ as they are proportional to the triangle symplectic area.

4.3. Lateral phases as relative phases

Let us write the vector $|A\rangle$ as

$$|A\rangle = \alpha|a\rangle + \beta|B\rangle + \gamma|C\rangle \tag{4.5}$$

with α, β, γ complex coefficients. This can always be done, since the three vectors $|A\rangle, |B\rangle, |C\rangle$ lie on a subspace of $\mathcal{H}$ spanned by ψ_a, Ψ_B, Ψ_C. The two coefficients β, γ in (4.5) parametrize the way the two *states* Ψ_B, Ψ_C enter the superposition Ψ_A (recall ψ_a, Ψ_B, Ψ_C is not an orthogonal basis; only ψ_a is orthogonal to Ψ_B, Ψ_C and hence this is not the conventional expansion in a basis). These parameters can also be expressed as

$$|A\rangle = \langle a, A\rangle|a\rangle + \frac{\sin b}{\sin a}\langle b, a\rangle|B\rangle + \frac{\sin c}{\sin a}\langle c, a\rangle|C\rangle. \tag{4.6}$$

The *relative phase* Θ^A_{BC} with which the *states* Ψ_B, Ψ_C enter the superposition Ψ_A must be independent of the choice of representant vectors $|A\rangle, |B\rangle, |C\rangle$; any phase change in them should be accompanied by a change in the coefficients β, γ in order to keep the vector $|A\rangle$ in the same ray as before, so the relative phase *is not simply equal to* the phase difference between the coefficients. The correct expression of Θ^A_{BC} giving the relative phase with which the *states* Ψ_B, Ψ_C enter the superposition Ψ_A (4.5) is

$$\Theta^A_{BC} = \arg(\langle\beta B, \gamma C\rangle) = \arg(\langle a, b\rangle\langle B, C\rangle\langle c, a\rangle). \tag{4.7}$$

Trigonometry allows us to interpret Θ^A_{BC} in terms of invariants of the triangle, Ψ_A, Ψ_B, Ψ_C, since the equation

$$\mathrm{e}^{-\mathrm{i}\phi_a} = \frac{\langle a, b\rangle\langle B, C\rangle\langle c, a\rangle}{|\langle a, b\rangle| \cdot |\langle B, C\rangle| \cdot |\langle c, a\rangle|} \tag{4.8}$$

from (2.17) identifies the relative phase Θ^A_{BC} with which the two states $|B\rangle, |C\rangle$ enter a superposition $|A\rangle$ with the opposite lateral phase $-\phi_a$ of the triangle. Note this equals the phase difference between the coefficients β, γ *only when the vector states* $|B\rangle, |C\rangle$ *are chosen so that they are in phase in the Pancharatnam sense*, for in this case $\langle B, C\rangle$ would be real and positive; this is the usual statement linking the relative phase with the phase difference between coefficients once the vector states have been chosen in a prescribed way.

4.4. Relative phases and superposition versus decomposition principle

All information relative to phases *can* of course be obtained by making reference to $\mathcal{H}$ also; this is carried out by the Pancharatnam proposal. The stress here concerns the fact that $\mathcal{H}$ can be, in principle, completely avoided: the *intrinsic* geometry of the space of states $\mathcal{P}$ contains all information on relative phases and this information can be retrieved without any reference to the ambient linear Hilbert space. The study of these intrinsic relations sheds some light on quantities which are physically meaningful. We emphasize that relative phases involve *three* states, and should *not* be looked at anyhow as *the relative phase between* Ψ_B, Ψ_C, an idea which for states is clearly meaningless. Rather, relative phases describe how the two states Ψ_B, Ψ_C 'appear' when decomposing a given third state Ψ_A; thus this is akin to considering the superposition principle as a 'decomposition principle', as stressed by Jauch and several other authors [18].

While the relative mutual positions of the three states are *not* completely described by their mutual distances, adding another independent invariant (either symplectic area or a single lateral phase) suffices to determine them up to an $SU(3)$ isometry; this corresponds to the complete specification of a physical state in term of transition probabilities *and* relative phases.

4.5. Projecting triangles on the $\mathbb{C}P^1$ containing a side and the complex collinear case

In terms of $\mathbb{C}P^2$, the state $[A]$ in (4.5) has a well-defined Hermitian orthogonal projection $[A]'$ over the $\mathbb{C}P^1$ submanifold determined by $[B]$, $[C]$, with representing vector $|\Psi_A\rangle'$:

$$|\Psi_A\rangle' = \beta|\Psi_B\rangle + \gamma|\Psi_C\rangle. \tag{4.9}$$

The relation between lateral phases and superposition of states suggests studying the triangle with vertices Ψ'_A, Ψ_B, Ψ_C corresponding to the projection of the original triangle over the (unique) complex line $\mathbb{C}P^1$ containing Ψ_B, Ψ_C. Denoting invariants for the projected triangle with primes, these are related to the original triangle invariants by

$$\begin{array}{lll} a' = a & b' \quad (\tan b' = \tan b \cos C) & c' \quad (\tan c' = \tan c \cos B) \\ A' = 0 & B' = 0 & C' = 0 \\ \phi'_a = \phi_a & \phi'_b = \phi_b - \omega & \phi'_c = \phi_c - \omega \\ \Phi'_A = \Phi_A - \omega & \Phi'_B = \Phi_B & \Phi'_C = \Phi_C \end{array} \tag{4.10}$$

and through (3.1) applied to (4.10), for the triangular phases of the projected triangle:

$$\Omega' = \Omega \qquad \omega' = 0. \tag{4.11}$$

Thus the projected triangle has Hermitian angles $A' = B' = C' = 0$ and $s' = 0$, while its *symplectic area* $\mathcal{S}'$ equals the initial one, as do the two invariants a', ϕ'_a and the two *angular phases* Φ_B, Φ_C. The relation $\phi'_a = \phi_a$ reflects the equality between the relative phases with which Ψ_A, Ψ_B enter into either superposition Ψ'_A or Ψ_A, thus confirming the identification of lateral phase in section 4.3. Another interesting relation is the equality of the angular phases at vertices $[B], [C]$ for the original and projected triangles; this is related to the conservation of symplectic area under Hermitian projection, because

$$\sin(2\mathcal{S}) = \frac{\sin\Phi_B \sin\Phi_C \sin^2 a}{\sin\phi_a} = \frac{\sin\Phi'_B \sin\Phi'_C \sin^2 a'}{\sin\phi'_a} = \sin(2\mathcal{S}'). \quad (4.12)$$

As a sideline complement, we may study triangles completely contained in a $\mathbb{C}P^1$, where the state Ψ_A is a superposition of Ψ_B, Ψ_C only. This occurs when for one vertex, say $[C]'$, the Hermitian angle at it has vanishing $\sin C'$ (in the following, to denote that we restrict to a complex line $\mathbb{C}P^1$ a prime will be appended to all trigonometric quantities; this is consistent with the previous choice in (4.10)). We may always assume $C' = 0$ as $(C' = \pi, \Phi'_C) \equiv (C' = 0, \Phi'_C + \pi)$. From (A.3) *all* Hermitian angles have vanishing sines, $\sin A' = \sin B' = \sin C' = 0$. Equations (A.35) imply $\sin\omega' = 0$, and (A.2), (3.6) reduce to

$$\phi'_a - \Phi'_A = \Phi'_B - \phi'_b = \Phi'_C - \phi'_c = \Omega' \qquad 2\Omega' = (-\Phi'_A + \Phi'_B + \Phi'_C). \quad (4.13)$$

Thus the essential invariants of a $\mathbb{C}P^1$ triangle are a', b', c'; $\Phi'_A, \Phi'_B, \Phi'_C$ as ω; A', B', C' all vanish and the remaining Ω; $\phi'_a, \phi'_b, \phi'_c$ can be expressed in terms of the former. From the basic equations one may check that a', b', c'; $\Phi'_A, \Phi'_B, \Phi'_C$ are the sides and angles of a spherical triangle in the sphere $S^2_{K=4}$ of radius $1/2$, whose Riemannian area $\mathcal{A}'$ is related to its ordinary angular excess by $4\mathcal{A}' = -\Phi'_A + \Phi'_B + \Phi'_C$. Using (4.13), we get $4\mathcal{A}' = 4\mathcal{S}'$, hence the symplectic area of a complex collinear triangle in $\mathbb{C}P^1$ coincides with the ordinary Riemannian area (with other normalizations these would only be proportional).

4.6. Some explicit examples

In this section we provide explicit examples displaying in the plain language of ordinary quantum mechanics our result concerning computation of geometric phases for triangle loops through the BTB invariant Ω; this result holds for *generic triangles in any* $\mathbb{C}P^N$. The setting is the one explained in subsection 4.2: we make a sequence of three filtering measurements on the initial state Ψ_A, which allow us to pass the system through the states Ψ_B, Ψ_C, Ψ_A; this makes the final vector state differ from the initial one by a phase $e^{i\Sigma}$. Situations with $N = 1$ (a two-dimensional $\mathcal{H}$) are well known: standard examples are a spin-$1/2$ particle or polarization of a monochromatic light plane wave, whose spaces of states are the sphere of spin-$1/2$ directions or the Poincaré sphere; North and South poles correspond to a fixed orthonormal basis $\{|1\rangle, |2\rangle\}$ (say $\{|+\rangle, |-\rangle\}$ for spin-$1/2$ or $\{|R\rangle, |L\rangle\}$ circular polarizations for light) and the point with geographical coordinates (θ, ϕ) corresponds to the state

$$|\Psi\rangle_{\theta,\phi} \propto \cos(\theta/2)|1\rangle + \sin(\theta/2)\, e^{i\phi}|2\rangle. \quad (4.14)$$

In this case, starting from the state Ψ_{θ_1,ϕ_1} we make three successive filtering measurements allowing us to pass the states corresponding to the points (θ_2, ϕ_2), (θ_3, ϕ_3) and finally (θ_1, ϕ_1) again, as in the original Pancharatnam setting. The state thus follows a closed loop on the sphere, the spherical triangle with vertices (θ_1, ϕ_1), (θ_2, ϕ_2), (θ_3, ϕ_3) and the shortest geodesics joining them. Let Φ_1, Φ_2, Φ_3 be the ordinary (inner) angles of this triangle; the geometrical phase associated with this triangle is equal to $\Sigma_{123}/2$ [1], where Σ_{123} is the solid angle of the geodesic spherical triangle 123 which on a sphere of any radius is equal to the ordinary triangle angular excess $\Phi_1 + \Phi_2 + \Phi_3 - \pi$.

Let us see how our approach includes this result as a very particular case, while affording many alternative ways to compute the triangle geometric phase Σ in *any* $\mathbb{C}P^N$. We will start by considering a generic triangle (that for any N fits in a $\mathbb{C}P^2$), and then will see how this generic result reduces for the simpler case of a two-level system, $N = 1$.

The general result for the geometric phase Σ associated with *any triangle*, as given by the BTB invariant, is

$$\Sigma = \Omega = -\phi_a + \Phi_B + \Phi_C = -\Phi_A + \phi_b + \Phi_C = -\Phi_A + \Phi_B + \phi_c \tag{4.15}$$

where the lateral and angular phases are linked through the general relations (see (3.3)):

$$\phi_a - \Phi_A = \Phi_B - \phi_b = \Phi_C - \phi_c = \Omega - \omega. \tag{4.16}$$

All trigonometric equations which involve Ω may be used to obtain alternative closed expressions for Σ. For instance, from the appendix (A.5), it follows that

$$\Sigma = \Omega = \arg(1 + \tan a \tan b \cos C\, \mathrm{e}^{\mathrm{i}\Phi_C}) = \frac{1}{2\mathrm{i}} \log \frac{1 + \tan a \tan b \cos C\, \mathrm{e}^{\mathrm{i}\Phi_C}}{1 + \tan a \tan b \cos C\, \mathrm{e}^{-\mathrm{i}\Phi_C}} \tag{4.17}$$

which gives the geometric phase in terms of the triangle side–angle-side data (two sides a, b and the two components—Hermitian angle C and angular phase Φ_C—of the complete angle at the vertex Ψ_C). From (A.34) we get another general formula involving data a, ϕ_a relative to a side and angular phases Φ_B, Φ_C at the two adjacent vertices:

$$\sin \Sigma = \frac{\sin \Phi_B \sin \Phi_C \sin^2 a}{\sin \phi_a}. \tag{4.18}$$

To relate with the usual formulation, let us denote by $|1\rangle, |2\rangle, |3\rangle$ the $\mathbb{C}^3$ orthonormal basis where the three vector states are

$$\begin{aligned} &|\Psi_C\rangle = |1\rangle \qquad |\Psi_B\rangle = \cos a|1\rangle + \sin a|2\rangle \\ &|\Psi_A\rangle = \cos b|1\rangle + \sin b\, \mathrm{e}^{\mathrm{i}\Phi_C}(\cos C|2\rangle + \sin C|3\rangle). \end{aligned} \tag{4.19}$$

In this adapted basis the vector components are directly related to the two sidelengths a, b and the two components of the complete angle at the vertex Ψ_C, and in terms of these data (4.17) provides a closed expression for the geometric phase Σ along the triangle. Again the remaining triangle invariants (e.g, other angular phases such as Φ_B or lateral phases such as ϕ_a) can be easily computed; the same procedure may also be used when the three initial vectors are given in any arbitrary, non-adapted, basis. The three polar vectors, as computed from (2.11), are

$$\begin{aligned} &|\psi_a\rangle = |0\rangle \qquad |\psi_b\rangle = \mathrm{e}^{-\mathrm{i}\Phi_C}(-\sin C|2\rangle + \cos C|3\rangle) \\ &|\psi_c\rangle = \frac{1}{\sin c}\{\sin b \sin C\, \mathrm{e}^{-\mathrm{i}\Phi_C}(-\sin a|1\rangle + \cos a|2\rangle) \\ &\qquad\qquad + (\sin a \cos b - \cos a \sin b \cos C\, \mathrm{e}^{-\mathrm{i}\Phi_C})|3\rangle\} \end{aligned} \tag{4.20}$$

where $\sin c \equiv (1 - \cos^2 a \cos^2 b - \sin^2 a \sin^2 b \cos^2 C - 2 \sin a \sin b \cos a \cos b \cos C \cos \Phi_C)^{1/2}$ is the normalizing factor. From these expressions by using (2.14) or (2.17) we may get any triangle invariant in terms of the vector components (here in terms of a, b, C, Φ_C). From angular or lateral phases we may compute the geometric phase as a mixed phase excess, or alternatively, from the first expression in (2.17) which gives the BTB invariant in terms of state vectors; this procedure also leads to (4.17) as the reader may check.

Let us now see what happens in the special case where all three states live on a $\mathbb{C}P^1$ (the three vector states span a $\mathbb{C}^2$ subspace of $\mathcal{H}$), which will be notationally distinguished by appending primes to the relevant quantities. For a triangle contained in a $\mathbb{C}P^1$ both the Hermitian angles and the dual BTB invariant vanish: $A' = B' = C' = 0, \omega' = 0$. From (A.2) the lateral phases *can* be expressed in terms of angular phases and Ω' alone by (4.13). But as the angular phases of the Hermitian triangle in $\mathbb{C}P^1$ are the ordinary angles of the spherical triangle (with our convention, $\Phi_A = \pi - \Phi_1$, $\Phi_B = \Phi_2$, $\Phi_C = \Phi_3$), this means that Ω' *itself* equals one half of the triangle ordinary angular excess $-\Phi'_A + \Phi'_B + \Phi'_C$. This is the well-known Pancharatnam result. When $C' \equiv C = 0$, (4.19) reduces to

$$|\Psi_C\rangle = |1\rangle \qquad |\Psi_B\rangle = \cos a'|1\rangle + \sin a'|2\rangle \qquad |\Psi_A\rangle = \cos b'|1\rangle + \sin b'\,\mathrm{e}^{\mathrm{i}\Phi'_C}|2\rangle \tag{4.21}$$

which after (4.14) corresponds to the three points on the sphere, with geographical coordinates $(0, 0)$ (North pole) for $|\Psi_C\rangle$, $(\theta_2 = 2a', 0)$ (placed on the reference meridian) for $|\Psi_B\rangle$ and $(\theta_3 = 2c', \phi_3 = \Phi'_C)$ for $|\Psi_A\rangle$; thus the two triangle sides are directly b', c' and the enclosed angle is Φ'_C. The geometric phase for this triangle may be computed alternatively as

$$\begin{aligned}\Sigma' = \Omega' = \tfrac{1}{2}(-\Phi'_A + \Phi'_B + \Phi'_C) &= \arg(1 + \tan a' \tan b'\,\mathrm{e}^{\mathrm{i}\Phi'_C}) \\ &= \frac{1}{2\mathrm{i}} \log \frac{1 + \tan a' \tan b'\,\mathrm{e}^{\mathrm{i}\Phi'_C}}{1 + \tan a' \tan b'\,\mathrm{e}^{-\mathrm{i}\Phi'_C}}\end{aligned} \tag{4.22}$$

whose consistence is a routine exercise in spherical trigonometry.

Thus the identification of the geometric phase Σ with an ordinary angular excess is specific to the $N = 1$ case. Its proper extension to the general case involves both *angular* and *lateral* phases in a kind of *mixed* phase excess in a expression $\Sigma = \Omega = -\phi_a + \Phi_B + \Phi_C$ whose simplicity is due to dealing with the new *phase* triangle invariants. This simplicity is lost once we describe the three states in terms of the components of their state vectors in some orthonormal basis, although the geometric phase may be computed from these expressions, as (4.17) shows.

Returning to the general expressions (in any N), we make a few remarks. First, as observed in the previous subsection, the BTB invariant Ω (and hence the geometric phase Σ) of a triangle does not change by Hermitian projection: this is embodied in (4.17) and (4.22), as the side-angle-side data a'; C', Φ'_C; b' of the projected triangle are related to the original ones as $a' = a$, $\tan b' = \tan b \cos C$, $C' = 0$, $\Phi'_C = \Phi_C$, which according to (4.17) and (4.22) confirms the equality $\Omega' = \Omega$. Second, for fixed a, b, Φ_C, the geometric phase Σ reaches its maximum value for $C = 0$ and vanishes for $C = \pi/2$; third, the geometric phase also vanishes if $\Phi_C = 0$ while $C \neq 0$. In an N-level system the triangle geometric phase behaviour is thus more complicated than in the $N = 1$ case, but can be adequately grasped through this analysis.

5. Further applications

5.1. A dictionary

The correspondence *dictionary* includes the geometric objects discussed and their physical interpretation. We include a few quantities whose geometrical meaning is clear but whose physical interpretation is not still settled.

Hermitian elliptic geometry	*versus*	Quantum physics
Hermitian elliptic space	•	Quantum space of states $\mathcal{P}$
Point $[C]$	•	State Ψ_C (ray in $\mathcal{H}$)
(Short) geodesic segment $[C] \to [B]$	•	Quantum evolution for a Pancharatnam-type filtering measurement $\Psi_C \to \Psi_B$
FS distance a between points $[C]$, $[B]$	•	Probability $\cos^2 a$ for finding Ψ_B in the state Ψ_C.
Lateral phase ϕ_a in triangle $[A]$, $[B]$, $[C]$	•	Relative phase between states Ψ_B, Ψ_C when Ψ_A is decomposed in terms of Ψ_B, Ψ_C
Hermitian angle A between geodesics	•	?? (but $A = 0$ is equivalent to Ψ_A *is a superposition of* Ψ_B, Ψ_C *alone*)
Angular phase Φ_A between geodesics	•	??
'Mixed' phase excess $\Omega \propto$ triangle symplectic area	•	Anandan–Aharonov geometric phase Σ for the triangle circuit.
Dual 'mixed' phase excess $\omega \propto$ triangle 'symplectic coarea'	•	?? (but $\omega = 0$ is equivalent to Ψ_A *is a superposition of* Ψ_B, Ψ_C *alone*)
Fubini–Study metric parallel transport in $\mathcal{P}$	•	Quantum parallel transport behind AA phase

5.2. Geophasics and null phase curves

When a closed circuit in the state space is obtained by joining the three states with geodesics, the AA geometric phase along this circuit is given by

$$\Sigma = 2\mathcal{S} = -\arg(\langle A, B\rangle\langle B, C\rangle\langle C, A\rangle). \tag{5.1}$$

Are there other types of curves which may replace geodesics but such that the previous statement stays true? This has been discussed by Rabei *et al* [24] who introduce 'null phase curves' in $\mathcal{P}$, and thus prove that these curves are the broadest class with the property that if three states $|A\rangle, |B\rangle, |C\rangle$ are joined by any such curves, the AA geometric phase along the closed circuit so obtained is still given by (5.1). Can we understand these null phase curves from our approach to trigonometry? The key idea is the two-component nature of the angles in $\mathcal{P}$. The analogue of Frenet–Serret equations for a curve $\Psi(l)$ in $\mathcal{P}$ would involve a *two-component* curvature, measuring the rate of change (relative to the arclength parameter l) of the *Hermitian angle* and the *angular phase* between the tangent vector and a suitable fiducial vector along the curve (obtained at the point $\Psi(l)$ by parallel transport of some reference tangent vector at an initial point, say $\Psi(l_0)$). Both quantities can be defined as the two components of the angle between the tangent vector to the curve at the point $\Psi(l)$ and its FS covariant derivative and may be called 'Hermitian curvature' and 'phase curvature' of the curve. The geodesics in $\mathbb{C}P^2$ are characterized among general curves in $\mathcal{P}$ by the vanishing of this *two-component curvature* at all points. Yet one may consider curves where just *one* of these components of the curvature vanishes. If the Hermitian curvature is set to zero, the resulting curve is contained in a $\mathbb{C}P^1$ submanifold yet it is not necessarily a geodesic (in fact, it may be *any curve in a* $\mathbb{C}P^1$). If only the phase curvature is set to zero, but no restriction is placed on the Hermitian curvature, the curve will not be in general a geodesic, but will still be clearly a distinguished curve. We propose to call these curves *geophasics* or *isophasics*. It would be interesting to ascertain to what extent these curves are related to the so-called null phase curves in [24].

5.3. Exponential and 'holonomy' identities

Each trigonometric invariant is naturally associated with a Lie algebra generator [3]. For instance, each side g_a has associated a pair of *commuting* 'pure translation' and 'phase translation' generators P_a, T_a in the Lie algebra $\mathfrak{su}(3)$ of the group $SU(3)$. Similarly, at each vertex, say C, there are two commuting generators of 'pure rotations' and 'phase rotations' J_C, I_C. With each vertex or side we can associate a 'complete rotation' around the vertex $\mathrm{e}^{CJ_C}\,\mathrm{e}^{\Phi_C I_C}$ or 'complete translation' along the side $\mathrm{e}^{aP_a}\,\mathrm{e}^{\phi_a T_a}$; each separate factor is also a group transformation leaving the vertex or side invariant.

Trigonometry for the quantum space of states may also be expressed through several 'loop identities' involving such exponentials associated with the triangle sides and vertices; their structure is quite intriguing [3, 25, 26]. As an example we write down a few where the four phase excesses Ω, Δ_Φ, ω, δ_ϕ appear (see (3.9) and [3] for further details on notation):

$$\begin{aligned}
&\mathrm{e}^{-aP_a}\,\mathrm{e}^{-\phi_a T_a}\,\mathrm{e}^{cP_c}\,\mathrm{e}^{\phi_c T_c}\,\mathrm{e}^{bP_b}\,\mathrm{e}^{\phi_b T_b} = \mathrm{e}^{-\Delta_\Phi I_C}\,\mathrm{e}^{-(-A+B+C)J_C}\\
&\mathrm{e}^{-AJ_A}\,\mathrm{e}^{-\Phi_A I_A}\,\mathrm{e}^{CJ_C}\,\mathrm{e}^{\Phi_C I_C}\,\mathrm{e}^{BJ_B}\,\mathrm{e}^{\Phi_B I_B} = \mathrm{e}^{-\delta_\phi T_c}\,\mathrm{e}^{-(-a+b+c)P_c}
\end{aligned} \tag{5.2}$$

$$\begin{aligned}
&\mathrm{e}^{-aP_a}\,\mathrm{e}^{cP_c}\,\mathrm{e}^{bP_b} = \mathrm{e}^{-(3/2)\Omega I_C}\{\mathrm{e}^{(\phi_a/2)B_C}\,\mathrm{e}^{-BJ_C}\,\mathrm{e}^{(-\phi_c/2)B_C}\,\mathrm{e}^{AJ_C}\,\mathrm{e}^{(-\Phi_b/2)B_C}\,\mathrm{e}^{-CJ_C}\}\\
&\mathrm{e}^{-AJ_A}\,\mathrm{e}^{CJ_C}\,\mathrm{e}^{BJ_B} = \mathrm{e}^{-(3/2)\omega T_c}\{\mathrm{e}^{(\Phi_A/2)H_c}\,\mathrm{e}^{-bP_c}\,\mathrm{e}^{(-\Phi_C/2)H_c}\,\mathrm{e}^{aP_c}\,\mathrm{e}^{(-\phi_B/2)H_c s}\,\mathrm{e}^{-cP_c}\}.
\end{aligned} \tag{5.3}$$

All these equations share a common structure. The left-hand sides are products of suitable 'triangular' transformations *looping* along the sides and vertices of the triangle, while the right-hand sides are 'fiducial' transformations, relative to a *single* vertex or side; in the form we display all these are either rotations/phase rotations around the vertex C or translations/phase translations along the side c.

The four equations displayed are grouped in two dual pairs, and this is underlined by the writing conventions: duality interchanges 'upper case' quantities A, Φ_A by 'lower case' ones a, ϕ_a, and their corresponding rotation generators J, I by translation ones P, T. To explore the meaning of these equations, let us consider the first equation in (5.3), which may be written as

$$\mathrm{e}^{-aP_a}\,\mathrm{e}^{cP_c}\,\mathrm{e}^{bP_b} = \mathrm{e}^{-(3/2)\Omega I_C}\cdot\mathrm{e}^{\hat{\Theta}\hat{\mathbb{J}}_C}. \tag{5.4}$$

To interpret the operator on the left-hand side, let us recall that the exponentials e^{cP_c} of the 'pure translation' generators P_c carry out the ordinary FS parallel transport on tangent vectors. Thus the product $\mathrm{e}^{-aP_a}\,\mathrm{e}^{cP_c}\,\mathrm{e}^{bP_b}$ is the FS parallel transport operator along the triangle, and (5.4) gives a closed expression for its holonomy operator as a product whose two factors correspond to the *two* commuting (semi)simple factors $U(1)\otimes SU(2)$ in the isotropy subgroup of the action of $SU(3)$ in $\mathcal{P}$; I_C generate the $U(1)$ part of the isotropy subgroup of the vertex $[C]$ and $\hat{\mathbb{J}}_C$ belongs to the Lie algebra of the $SU(2)$ part. If the vertex $[C]$ of the triangle is considered to be located at the origin, (5.4) can be written in the isotropy representation at the origin. The full expression for $\hat{\Theta}\hat{\mathbb{J}}_C$, which can be read from (5.3), is uninteresting for our purposes. However, the expression for I_C in this representation is very simple,

$$I_C \to \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \tag{5.5}$$

and therefore the $U(1)$ part of the holonomy associated with the parallel transport in $\mathcal{P}$ along the triangle has a holonomy phase angle proportional to Ω. When (5.4) is considered for the N-dimensional space $SU(N+1)/U(1)\otimes SU(N)$, it changes to

$$\mathrm{e}^{-aP_a}\,\mathrm{e}^{cP_c}\,\mathrm{e}^{bP_b} = \mathrm{e}^{\frac{-(N+1)}{N}\Omega I_C}\cdot\mathrm{e}^{\hat{\Theta}\hat{J}_C} \tag{5.6}$$

so the proportionality factor $3/2$ is actually $(N+1)/N$, tending towards 1 as $N\to\infty$.

Another standard geometrical interpretation for the geometric phases [5] is as the holonomy transformation due to parallel transport in $S_{\mathcal{H}}$ (as a fibre bundle over $\mathcal{P}$) which translates vector states along the curves in $\mathcal{P}$ so that they are 'in phase'. Note that within this interpretation the operator $\mathrm{e}^{-aP_a}\,\mathrm{e}^{cP_c}\,\mathrm{e}^{bP_b}$ in (5.6), when considered as acting on the (ambient) linear Hilbert space $\mathcal{H}$ through the usual vector representation of $SU(3)$, is precisely this 'parallel transport operator'; in particular, this has been used in [16] to obtain an expression for the geometrical phase associated with an infinitesimal triangle. In the alternative interpretation, (5.4) is (also) the natural parallel transport operator that translates *tangent vectors* to $\mathcal{P}$ along the triangle, and then Ω is interpreted as the $U(1)$ part of the FS parallel transport holonomy in $\mathcal{P}$, with no reference whatsoever to $\mathcal{H}$.

5.4. Trigonometry and coherent states

Another application would be to study the trigonometry in submanifolds of coherent states in the space of states. For the usual harmonic oscillator coherent states, the symplectic area appears *directly* as the phase of the overlap of two coherent states $|z_A\rangle$, $|z_B\rangle$ associated with two vertices of the triangle, the third one having been taken as the fiducial origin $|z_C = 0\rangle$ [27]. The FS geometry of several families of coherent states is known [24, 28]; it would be interesting to relate this to the 'mixed phase excess' Ω of a triangle in the corresponding submanifold, whose intrinsic FS geometry does not come from a *linear* subspace of $\mathcal{H}$.

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Appendix A

A.1. A bestiary of trigonometric equations in $\mathcal{P}$

In section 2 we have given a complete set (3.1) and (3.2) of basic trigonometric equations. Starting from these, a complete bestiary can be obtained. These equations also follow by particularization of the 'general' CKD form given in [3] where each relation represents a trigonometry equation for each of the 27 spaces belonging to the CKD family, according to the values of the Cayley–Dickson η or Cayley–Klein parameters κ_1, κ_2; for the case under consideration these are all positive (and can be rescaled to 1). Besides, in [3] each such equation represented three trigonometric equations associated with three vertices or sides, according to a compact notation which implicitly included the signs associated with the vertex A and side a. Here we now list all these derived equations using the standard notation and displaying explicitly the signs in A and a. A few equations allow a clear writing in terms of alternative choices of angular invariants, at the price of losing the manifest self-duality. For instance, at the vertex C, the holomorphy inclination of the triangle is

$$\Upsilon_C = \arccos\left(\frac{\mathrm{Re}\langle \mathrm{i}u_{CA}, w\rangle}{\|u_{CA}\|\cdot\|w\|}\right) \quad \text{when} \quad \mathrm{Re}\langle u_{CA}, w\rangle = 0 \quad \mathrm{span}(u_{CA}, u_{CB}) = \mathrm{span}(u_{CA}, w). \tag{A.1}$$

which is related to other invariants at vertex C by equations similar to (2.10). For the sake of completeness, we have included a few of these rewritings. The relations between the different angular invariants at each vertex are collected together in (2.10).

• The Hermitian phases theorem:

$$-\Phi_A + \phi_a = \Phi_B - \phi_b = \Phi_C - \phi_c = \Omega - \omega. \tag{A.2}$$

• The Hermitian sine theorem

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \tag{A.3}$$

can be derived either from the complex Hermitian cosine theorem or from its dual. This self-dual relation is formally identical to the sine theorem of real spherical trigonometry. When expressed in terms of Λ, Υ at each vertex, it reads

$$\frac{\sin a}{\sin \Upsilon_A \sin \Lambda_A} = \frac{\sin b}{\sin \Upsilon_B \sin \Lambda_B} = \frac{\sin c}{\sin \Upsilon_C \sin \Lambda_C}. \tag{A.4}$$

• The Hermitian cosine theorem (for sides):

$$\begin{aligned}
\cos a\, e^{i\Omega} &= \cos b \cos c - \sin b \sin \cos A\, e^{-i\Phi_A} \\
\cos b\, e^{i\Omega} &= \cos c \cos a + \sin c \sin a \cos B\, e^{i\Phi_B} \\
\cos c\, e^{i\Omega} &= \cos a \cos b + \sin a \sin b \cos C\, e^{i\Phi_C}
\end{aligned} \tag{A.5}$$

and their dual Hermitian cosine laws (for angles):

$$\begin{aligned}
\cos A\, e^{i\omega} &= \cos B \cos C - \sin B \sin C \cos a\, e^{-i\phi_a} \\
\cos B\, e^{i\omega} &= \cos C \cos A + \sin C \sin A \cos b\, e^{i\phi_b} \\
\cos C\, e^{i\omega} &= \cos A \cos B + \sin A \sin B \cos c\, e^{i\phi_c}.
\end{aligned} \tag{A.6}$$

• By equating the moduli of both sides of the Hermitian cosine theorem we get

$$\begin{aligned}
\cos^2 a &= (\cos b \cos c - \sin b \sin c \cos A \cos \Phi_A)^2 + \sin^2 b \sin^2 c \cos^2 A \sin^2 \Phi_A \\
\cos^2 b &= (\cos c \cos a + \sin c \sin a \cos B \cos \Phi_B)^2 + \sin^2 c \sin^2 a \cos^2 B \sin^2 \Phi_B \\
\cos^2 c &= (\cos a \cos b + \sin a \sin b \cos C \cos \Phi_C)^2 + \sin^2 a \sin^2 b \cos^2 C \sin^2 \Phi_C.
\end{aligned} \tag{A.7}$$

This is the Shirokov–Rosenfeld cosine theorem (A.11), see [10], yet expressed in terms of the angular variables A and Φ_A. This admits another form, starting from $\cos(2a) + 1 = 2\cos^2 a$, substituting (A.7) and expanding the squared sines of sides:

$$\begin{aligned}
\cos(2a) &= \cos(2b)\cos(2c) - \sin(2b)\sin(2c)\cos A \cos \Phi_A - 2\sin^2 b \sin^2 c \sin^2 A \\
\cos(2b) &= \cos(2c)\cos(2a) + \sin(2c)\sin(2a)\cos B \cos \Phi_B - 2\sin^2 c \sin^2 a \sin^2 B \\
\cos(2c) &= \cos(2a)\cos(2b) + \sin(2a)\sin(2b)\cos C \cos \Phi_C - 2\sin^2 a \sin^2 b \sin^2 C
\end{aligned} \tag{A.8}$$

which is the Shirokov–Rosenfeld cosine double theorem (A.12) in terms of the Hermitian angles and phases. Its dual theorems, not given by SR, may be called the Shirokov–Rosenfeld cosine theorem (for angles):

$$\begin{aligned}
\cos^2 A &= (\cos B \cos C - \sin B \sin C \cos a \cos \phi_a)^2 + \sin^2 B \sin^2 C \cos^2 a \sin^2 \phi_a \\
\cos^2 B &= (\cos C \cos A + \sin C \sin A \cos b \cos \phi_b)^2 + \sin^2 C \sin^2 A \cos^2 b \sin^2 \phi_b \\
\cos^2 C &= (\cos A \cos B + \sin A \sin B \cos c \cos \phi_c)^2 + \sin^2 A \sin^2 B \cos^2 c \sin^2 \phi_c
\end{aligned} \tag{A.9}$$

and the Shirokov–Rosenfeld dual cosine double theorem for angles:

$$\begin{aligned}
\cos(2A) &= \cos(2B)\cos(2C) - \sin(2B)\sin(2C)\cos a \cos \phi_a - 2\sin^2 B \sin^2 C \sin^2 a \\
\cos(2B) &= \cos(2C)\cos(2A) + \sin(2C)\sin(2A)\cos b \cos \phi_b - 2\sin^2 C \sin^2 A \sin^2 b \\
\cos(2C) &= \cos(2A)\cos(2B) + \sin(2A)\sin(2B)\cos c \cos \phi_c - 2\sin^2 A \sin^2 B \sin^2 c.
\end{aligned} \tag{A.10}$$

These SR cosine and cosine double theorems for sides were originally found [10] in terms of the pairs Υ_A, Λ_A of angular invariants, instead of A, ϕ_A:

$$\cos^2 a = (\cos b\cos c - \sin b\sin c\cos\Lambda_A)^2 + \sin^2 b\sin^2 c\cos^2\Upsilon_A\sin^2\Lambda_A$$
$$\cos^2 b = (\cos c\cos a + \sin c\sin a\cos\Lambda_B)^2 + \sin^2 c\sin^2 a\cos^2\Upsilon_B\sin^2\Lambda_B \qquad (A.11)$$
$$\cos^2 c = (\cos a\cos b + \sin a\sin b\cos\Lambda_C)^2 + \sin^2 a\sin^2 b\cos^2\Upsilon_C\sin^2\Lambda_C$$

$$\cos(2a) = \cos(2b)\cos(2c) - \sin(2b)\sin(2c)\cos\Lambda_A - 2\sin^2 b\sin^2 c\sin^2\Upsilon_A\sin^2\Lambda_A$$
$$\cos(2b) = \cos(2c)\cos(2a) + \sin(2c)\sin(2a)\cos\Lambda_B - 2\sin^2 c\sin^2 a\sin^2\Upsilon_B\sin^2\Lambda_B$$
$$\cos(2c) = \cos(2a)\cos(2b) + \sin(2a)\sin(2b)\cos\Lambda_C - 2\sin^2 a\sin^2 b\sin^2\Upsilon_C\sin^2\Lambda_C. \qquad (A.12)$$

• By building up the term $\sin(2b)\sin(2c)\cos A\cos\Phi_A$ in (A.5), substituting into (A.8), expanding and simplifying, we obtain

$$\cos^2 a = -\cos^2 b\cos^2 c + \sin^2 b\sin^2 c\cos^2 A + 2\cos a\cos b\cos c\cos\Omega$$
$$\cos^2 b = -\cos^2 c\cos^2 a + \sin^2 c\sin^2 a\cos^2 B + 2\cos a\cos b\cos c\cos\Omega \qquad (A.13)$$
$$\cos^2 c = -\cos^2 a\cos^2 b + \sin^2 a\sin^2 b\cos^2 C + 2\cos a\cos b\cos c\cos\Omega$$

which is the Blaschke–Terheggen cosine theorem for sides [7, 9]. Its dual is

$$\cos^2 A = -\cos^2 B\cos^2 C + \sin^2 B\sin^2 C\cos^2 a + 2\cos A\cos B\cos C\cos\omega$$
$$\cos^2 B = -\cos^2 C\cos^2 A + \sin^2 C\sin^2 A\cos^2 b + 2\cos A\cos B\cos C\cos\omega \qquad (A.14)$$
$$\cos^2 C = -\cos^2 A\cos^2 B + \sin^2 A\sin^2 B\cos^2 c + 2\cos A\cos B\cos C\cos\omega.$$

• By multiplying both sides of (A.3) by $1/\sin\Omega$ and using the second equation in (A.5) we obtain the Shirokov–Rosenfeld double sine theorem

$$\frac{\sin(2a)}{\sin\Phi_A\cos A} = \frac{\sin(2b)}{\sin\Phi_B\cos B} = \frac{\sin(2c)}{\sin\Phi_C\cos C} \qquad (A.15)$$

originally given [10] in terms of the angular invariants Υ, Λ:

$$\frac{\sin(2a)}{\cos\Upsilon_A\sin\Lambda_A} = \frac{\sin(2b)}{\cos\Upsilon_B\sin\Lambda_B} = \frac{\sin(2c)}{\cos\Upsilon_C\sin\Lambda_C}. \qquad (A.16)$$

Its dual is

$$\frac{\sin(2A)}{\sin\phi_a\cos a} = \frac{\sin(2B)}{\sin\phi_b\cos b} = \frac{\sin(2C)}{\sin\phi_c\cos c}. \qquad (A.17)$$

• By multiplying (A.15) and (A.17), we get a self-dual equation:

$$\frac{\sin a\sin A}{\sin\phi_a\sin\Phi_A} = \frac{\sin b\sin B}{\sin\phi_b\sin\Phi_B} = \frac{\sin b\sin C}{\sin\phi_c\sin\Phi_C}. \qquad (A.18)$$

• By taking the quotient between the double sine (A.15) and sine (A.3) theorems, we get

$$\frac{\cos a\tan A}{\sin\Phi_A} = \frac{\cos b\tan B}{\sin\Phi_B} = \frac{\cos c\tan C}{\sin\Phi_C}, \qquad (A.19)$$

which is very simple [10] in terms of Υ, Λ:

$$\cos a\tan\Upsilon_A = \cos b\tan\Upsilon_B = \cos c\tan\Upsilon_C. \qquad (A.20)$$

Its dual is

$$\frac{\cos A\tan a}{\sin\phi_a} = \frac{\cos B\tan b}{\sin\phi_b} = \frac{\cos C\tan c}{\sin\phi_c}. \qquad (A.21)$$

• Other equations are

$$\sin a\cos B\,\mathrm{e}^{\mathrm{i}\phi_c} = \cos b\sin c\,\mathrm{e}^{\mathrm{i}\Phi_A} + \sin b\cos c\cos A$$
$$-\sin b\cos C\,\mathrm{e}^{-\mathrm{i}\phi_a} = -\cos c\sin a\,\mathrm{e}^{-\mathrm{i}\Phi_B} + \sin c\cos a\cos B \qquad (A.22)$$
$$-\sin c\cos A\,\mathrm{e}^{\mathrm{i}\phi_b} = \cos a\sin b\,\mathrm{e}^{-\mathrm{i}\Phi_C} - \sin a\cos b\cos C$$

whose duals are

$$\begin{aligned}
\sin A \cos b \, e^{i\Phi_C} &= \cos B \sin C \, e^{i\phi_a} + \sin B \cos C \cos a \\
-\sin B \cos c \, e^{-i\Phi_A} &= -\cos C \sin A \, e^{-i\phi_b} + \sin C \cos A \cos b \\
-\sin C \cos a \, e^{i\Phi_B} &= \cos A \sin B \, e^{-i\phi_c} - \sin A \cos B \cos c
\end{aligned} \tag{A.23}$$

as well as the two sets of self-dual equations:

$$\begin{aligned}
\sin A \sin B + \cos A \cos B \cos c \, e^{i\phi_c} &= \sin a \sin b + \cos a \cos b \cos C \, e^{i\Phi_C} \\
-\sin B \sin C + \cos B \cos C \cos a \, e^{-i\phi_a} &= -\sin b \sin c + \cos b \cos c \cos A \, e^{-i\Phi_A} \\
\sin C \sin A + \cos C \cos A \cos b \, e^{i\phi_b} &= \sin c \sin a + \cos c \cos a \cos B \, e^{i\Phi_B}
\end{aligned} \tag{A.24}$$

$$\begin{aligned}
\cos A \cos B \cos c \sin \phi_c &= \cos a \cos b \cos C \sin \Phi_C \\
\cos B \cos C \cos a \sin \phi_a &= \cos b \cos c \cos A \sin \Phi_A \\
\cos C \cos A \cos b \sin \phi_b &= \cos c \cos a \cos B \sin \Phi_B .
\end{aligned} \tag{A.25}$$

• Starting from the real and imaginary parts of the 'complex Hermitian' cosine theorem (A.5), expanding the trigonometric functions of Ω by considering it as a sum of two phases, say $\Omega = (-\Phi_A) + (\phi_b + \Phi_C)$, and eliminating the term containing $\cos(\phi_b + \Phi_C)$ (or similarly for other splittings in (3.1)), we get

$$\begin{aligned}
\frac{\cos a}{\sin \Phi_A} &= \frac{\cos b \cos c}{\sin(\phi_b + \Phi_C)} = \frac{\cos b \cos c}{\sin(\Omega + \Phi_A)} \\
-\frac{\cos b}{\sin \Phi_B} &= \frac{\cos c \cos a}{\sin(\phi_c - \Phi_A)} = \frac{\cos c \cos a}{\sin(\Omega - \Phi_B)} \\
-\frac{\cos c}{\sin \Phi_C} &= \frac{\cos a \cos b}{\sin(-\phi_a + \Phi_B)} = \frac{\cos a \cos b}{\sin(\Omega - \Phi_C)} .
\end{aligned} \tag{A.26}$$

Its dual is

$$\begin{aligned}
\frac{\cos A}{\sin \phi_a} &= \frac{\cos B \cos C}{\sin(\Phi_B + \phi_c)} = \frac{\cos B \cos C}{\sin(\omega + \phi_a)} \\
-\frac{\cos B}{\sin \phi_b} &= \frac{\cos C \cos A}{\sin(\Phi_C - \phi_a)} = \frac{\cos C \cos A}{\sin(\omega - \phi_b)} \\
-\frac{\cos C}{\sin \phi_c} &= \frac{\cos A \cos B}{\sin(-\Phi_A + \phi_b)} = \frac{\cos A \cos B}{\sin(\omega - \phi_c)}
\end{aligned} \tag{A.27}$$

where we have used the relations $\Phi_B + \phi_c = \Omega - \Phi_A = \omega - \phi_a$ which follow from the equations in the 'Cartan' sector and the definitions of Ω and ω.

• By dividing equation (A.19) by (A.26), we get either

$$\begin{aligned}
\frac{\tan A}{\sin \Phi_A} &= \frac{-\tan C \cos b}{\sin(-\Phi_A + \phi_b)} = \frac{-\tan C \cos b}{\sin(\Omega - \Phi_C)} \\
\frac{\tan B}{\sin \Phi_B} &= \frac{\tan A \cos c}{\sin(\Phi_B + \phi_c)} = \frac{\tan A \cos c}{\sin(\Omega + \Phi_A)} \\
\frac{\tan C}{\sin \Phi_C} &= \frac{-\tan B \cos a}{\sin(\Phi_C - \phi_a)} = \frac{-\tan B \cos a}{\sin(\Omega - \Phi_B)}
\end{aligned} \tag{A.28}$$

or

$$\begin{aligned}
\frac{\tan A}{\sin \Phi_A} &= \frac{-\tan B \cos c}{\sin(-\Phi_A + \phi_c)} = \frac{-\tan B \cos c}{\sin(\Omega - \Phi_B)} \\
\frac{\tan B}{\sin \Phi_B} &= \frac{-\tan C \cos a}{\sin(\Phi_B - \phi_a)} = \frac{-\tan C \cos a}{\sin(\Omega - \Phi_C)} \\
\frac{\tan C}{\sin \Phi_C} &= \frac{\tan A \cos b}{\sin(\Phi_C + \phi_b)} = \frac{\tan A \cos b}{\sin(\Omega + \Phi_A)} .
\end{aligned} \tag{A.29}$$

The duals of these equations are

$$\frac{\tan a}{\sin\phi_a} = \frac{-\tan c\cos B}{\sin(-\phi_a+\Phi_B)} = \frac{-\tan c\cos B}{\sin(\omega-\phi_c)}$$
$$\frac{\tan b}{\sin\phi_b} = \frac{\tan a\cos C}{\sin(\phi_b+\Phi_C)} = \frac{\tan a\cos C}{\sin(\omega+\phi_a)} \qquad \text{(A.30)}$$
$$\frac{\tan c}{\sin\phi_c} = \frac{-\tan b\cos A}{\sin(\phi_c-\Phi_A)} = \frac{-\tan b\cos A}{\sin(\omega-\phi_b)}$$

$$\frac{\tan a}{\sin\phi_a} = \frac{-\tan c\cos C}{\sin(-\phi_a+\Phi_C)} = \frac{-\tan c\cos C}{\sin(\omega-\phi_b)}$$
$$\frac{\tan b}{\sin\phi_b} = \frac{\tan a\cos A}{\sin(\phi_b-\Phi_A)} = \frac{\tan a\cos A}{\sin(\omega-\phi_c)} \qquad \text{(A.31)}$$
$$\frac{\tan c}{\sin\phi_c} = \frac{-\tan b\cos B}{\sin(\phi_c+\Phi_B)} = \frac{-\tan b\cos B}{\sin(\omega+\phi_a)}.$$

• By eliminating the angles A, B, C using appropriately (A.29) and (A.28), we get

$$\cos^2 a = \frac{\sin(\Phi_B-\phi_a)\sin(\Phi_C-\phi_a)}{\sin\Phi_B\sin\Phi_C} = \frac{\sin(\Omega-\Phi_B)\sin(\Omega-\Phi_C)}{\sin\Phi_B\sin\Phi_C}$$
$$\cos^2 b = \frac{\sin(\Phi_C+\phi_b)\sin(-\Phi_A+\phi_b)}{-\sin\Phi_C\sin\Phi_A} = \frac{\sin(\Omega-\Phi_C)\sin(\Omega+\Phi_A)}{-\sin\Phi_C\sin\Phi_A} \qquad \text{(A.32)}$$
$$\cos^2 c = \frac{\sin(-\Phi_A+\phi_c)\sin(\Phi_B+\phi_c)}{-\sin\Phi_A\sin\Phi_B} = \frac{\sin(\Omega+\Phi_A)\sin(\Omega-\Phi_B)}{-\sin\Phi_A\sin\Phi_B}$$

whose duals are

$$\cos^2 A = \frac{\sin(\phi_b-\Phi_A)\sin(\phi_c-\Phi_A)}{\sin\phi_b\sin\phi_c} = \frac{\sin(\omega-\phi_b)\sin(\omega-\phi_c)}{\sin\phi_b\sin\phi_c}$$
$$\cos^2 B = \frac{\sin(\phi_c+\Phi_B)\sin(-\phi_a+\Phi_B)}{-\sin\phi_c\sin\phi_a} = \frac{\sin(\omega-\phi_c)\sin(\omega+\phi_a)}{-\sin\phi_c\sin\phi_a} \qquad \text{(A.33)}$$
$$\cos^2 C = \frac{\sin(-\phi_a+\Phi_C)\sin(\phi_b+\Phi_C)}{-\sin\phi_a\sin\phi_b} = \frac{\sin(\omega+\phi_a)\sin(\omega-\phi_b)}{-\sin\phi_a\sin\phi_b}.$$

These equations somehow resemble the real Euler's equations for the cosine of half the sides (angles) in terms of angles (sides) of a spherical triangle. In these Hermitian 'Euler-like' equations, pure sides (angles) are instead given in terms of angular (lateral) *phases*.

• By the expansion of sines of sums or differences and elementary manipulation, we finally get the expression for the squared sines of the sides in terms of phases *alone*:

$$\sin^2 a = \frac{\sin\phi_a\sin\Omega}{\sin\Phi_B\sin\Phi_C} \qquad \sin^2 b = \frac{\sin\phi_b\sin\Omega}{\sin\Phi_C\sin\Phi_A} \qquad \sin^2 c = \frac{\sin\phi_c\sin\Omega}{\sin\Phi_A\sin\Phi_B} \qquad \text{(A.34)}$$

whose dual equations are

$$\sin^2 A = \frac{\sin\Phi_A\sin\omega}{\sin\phi_b\sin\phi_c} \qquad \sin^2 B = \frac{\sin\Phi_B\sin\omega}{\sin\phi_c\sin\phi_a} \qquad \sin^2 C = \frac{\sin\Phi_C\sin\omega}{\sin\phi_a\sin\phi_b}. \qquad \text{(A.35)}$$

A.2. The FS metric and the symplectic structure of $\mathcal{P}$ from its trigonometry

As an example of the basic nature of trigonometry as part of the intrinsic geometry of $\mathcal{P}$ (just as for any other space), we will now directly obtain explicit expressions for the two basic

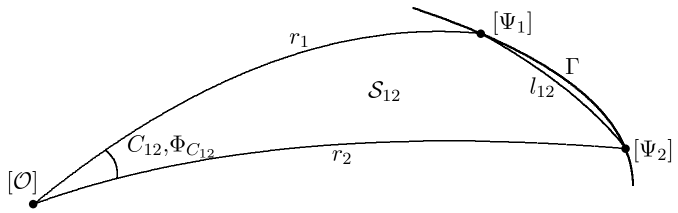

Figure 6. Triangle formed by the two states $[\Psi_1]$, $[\Psi_2]$ along a curve $[\Psi(t)]$ and the origin.

structures in $\mathcal{P}$, taking as the only input the (appropriate) trigonometric equations. Let us first consider the following parametrization for state vectors in $\mathbb{C}^{N+1}$:

$$[\Psi] = [\Psi(\theta, \alpha)] = \left[\begin{pmatrix} \cos(r) \\ \sin(r)\cos(\theta^2)\,\mathrm{e}^{\mathrm{i}\alpha^1} \\ \sin(r)\sin(\theta^2)\cos(\theta^3)\,\mathrm{e}^{\mathrm{i}(\alpha^1+\alpha^2)} \\ \vdots \\ \sin(r)\cdots\sin(\theta^{N-1})\cos(\theta^N)\,\mathrm{e}^{\mathrm{i}(\alpha^1+\cdots+\alpha^{N-1})} \\ \sin(r)\cdots\sin(\theta^{N-1})\sin(\theta^N)\,\mathrm{e}^{\mathrm{i}(\alpha^1+\cdots+\alpha^{N-1}+\alpha^N)} \end{pmatrix}\right] \tag{A.36}$$

in terms of $2N$ coordinates (θ^i, α^i), $i = 1, \ldots, N$; for θ^1 we use the special name $r \equiv \theta^1$. We will call (A.36) the geodesic polar parametrization of the associated $\mathbb{C}P^N$, because it reduces for the purely real submanifold $\alpha^i = 0, \pi$ to the well-known geodesic polar parametrization for the real S^N covering twice the real projective subspace $\mathbb{R}P^N$.

Let us now consider any curve Γ in $\mathcal{P}$, which can be described in terms of a parameter, say t, as $t \to \Psi(t) = \Psi(\theta(t), \alpha(t))$. For any two points $[\Psi_1]$, $[\Psi_2]$ on the curve, with the parameters t_1, t_2, let us consider the triangle with vertices $[\mathcal{O}] = [(1\ 0\ldots 0)]$, $[\Psi_1]$ and $[\Psi_2]$ (see figure 6). Its elements are related by trigonometric equations, and the cosine theorem (3.2) gives

$$\cos(l_{12})\,\mathrm{e}^{\mathrm{i}2\mathcal{S}_{12}} = \cos r_1 \cos r_2 + \sin r_1 \sin r_2 \cos(C_{12})\,\mathrm{e}^{\mathrm{i}\Phi_{C12}}. \tag{A.37}$$

From this expression we can obtain both quantities l_{12}, $\mathcal{S}_{12}$ in terms of the vector states Ψ_1, Ψ_2. Equation (2.3) gives the tangent vectors to the two geodesic sides meeting at the origin, and then an equation similar to (2.4) gives the two-component ('complex') angle C_{12}, Φ_{C12}:

$$\begin{aligned} \cos C_{12}\,\mathrm{e}^{\mathrm{i}\Phi_{C12}} &= \frac{\langle u_{01}, u_{02}\rangle}{\|u_{01}\|\cdot\|u_{02}\|} \\ &= \cos\theta_1^2 \cos\theta_2^2\,\mathrm{e}^{\mathrm{i}(\alpha_2^1-\alpha_1^1)} + \sin\theta_1^2 \sin\theta_2^2 \cos\theta_1^3 \cos\theta_2^3\,\mathrm{e}^{\mathrm{i}((\alpha_2^1-\alpha_1^1)+(\alpha_2^2-\alpha_1^2))} + \cdots \\ &\quad + \sin\theta_1^2 \sin\theta_2^2 \cdots \sin\theta_1^{(N-1)} \sin\theta_2^{(N-1)} \cos\theta_1^N \cos\theta_2^N\,\mathrm{e}^{\mathrm{i}((\alpha_2^1-\alpha_1^1)+\cdots+(\alpha_2^{N-1}-\alpha_1^{N-1}))} \\ &\quad + \sin\theta_1^2 \sin\theta_2^2 \cdots \cos\theta_1^{(N-1)} \cos\theta_2^{(N-1)} \sin\theta_1^N \sin\theta_2^N\,\mathrm{e}^{\mathrm{i}((\alpha_2^1-\alpha_1^1)+\cdots+(\alpha_2^N-\alpha_1^N))}. \end{aligned} \tag{A.38}$$

When t_1, t_2 are nearby, say $t_1 = t$, $t_2 = t + \mathrm{d}t$, these reduce to *infinitesimal* expressions for the distance $\mathrm{d}l$ between the nearby points $[\Psi]$ and $[\Psi + \mathrm{d}\Psi] = [\Psi(\theta + \mathrm{d}\theta, \alpha + \mathrm{d}\alpha)]$ in $\mathcal{P}$ (i.e. the *metric*) and the symplectic area of the triangle (i.e. the symplectic structure). From these we can get the lengths along Γ and symplectic areas of *any* surface bounded by Γ by integrating $\mathrm{d}l$ and $\mathrm{d}\mathcal{S}$. For the distance $\mathrm{d}l$ from the modulus of (A.37), we get

$$\begin{aligned} \cos^2(\mathrm{d}l) &= \cos^2 r \cos^2(r + \mathrm{d}r) + \sin^2 r \sin^2(r + \mathrm{d}r) \cos^2(\mathrm{d}C) \\ &\quad + 2 \sin r \cos r \sin(r + \mathrm{d}r) \cos(r + \mathrm{d}r) \cos(\mathrm{d}C) \cos(\mathrm{d}\Phi_C) \end{aligned} \tag{A.39}$$

and by inserting (A.38) in (A.39) and taking the approximation up to *second* order, we get an explicit expression of the Riemannian (FS) metric at $[\Psi]$ in geodesic polar coordinates in $\mathbb{C}P^N$:

$$\mathrm{d}l^2 = (\mathrm{d}\theta^1)^2 + \sum_{j=2}^{N}\left(\prod_{s=1}^{j-1}\sin^2(\theta^s)\right)(\mathrm{d}\theta^j)^2 + \sum_{j=1}^{N}\left(\prod_{s=1}^{j}\sin^2(\theta^s)\left(1-\prod_{s=1}^{j}\sin^2(\theta^s)\right)\right)(\mathrm{d}\alpha^j)^2$$
$$+2\sum_{i\neq j}\left(\prod_{s=1}^{j}\sin^2(\theta^s)\left(1-\prod_{s=1}^{i}\sin^2(\theta^s)\right)\right)\mathrm{d}\alpha^i\,\mathrm{d}\alpha^j. \tag{A.40}$$

For the symplectic area, from the imaginary part of (A.37)

$$\cos(\mathrm{d}l)\sin(2\mathrm{d}\mathcal{S}) = \sin r\,\sin(r+\mathrm{d}r)\cos(\mathrm{d}C)\sin(\mathrm{d}\Phi_C) \tag{A.41}$$

we can easily obtain the differential relation between $\mathrm{d}\mathcal{S}$ and $\mathrm{d}\Phi_C$:

$$2\mathrm{d}\mathcal{S} = \sin^2(r)\,\mathrm{d}\Phi_C. \tag{A.42}$$

Besides, inserting (A.38) in (A.41) and taking the approximation up to *first* order, we get an explicit expression of the symplectic area of the triangle:

$$2\mathrm{d}\mathcal{S} = \sin^2(r)(\mathrm{d}\alpha^1 + \sin^2(\theta^2)\,\mathrm{d}\alpha^2 + \sin^2(\theta^2)\sin^2(\theta^3)\,\mathrm{d}\alpha^3 + \cdots$$
$$+ \sin^2(\theta^2)\sin^2(\theta^3)\cdots\sin^2(\theta^N)\,\mathrm{d}\alpha^N). \tag{A.43}$$

In the particular case $N = 2$, the expressions for the FS metric and the symplectic area of the triangle in terms of the parametrization of $\mathcal{P}$,

$$[\Psi] = \left[\begin{pmatrix}\cos(r)\\ \sin(r)\cos(\gamma)\,\mathrm{e}^{\mathrm{i}\phi}\\ \sin(r)\sin(\gamma)\,\mathrm{e}^{\mathrm{i}(\phi+\varphi)}\end{pmatrix}\right] \tag{A.44}$$

are

$$\mathrm{d}l^2 = \mathrm{d}r^2 + \sin^2 r\,\mathrm{d}\gamma^2 + \tfrac{1}{4}\sin^2(2r)\,\mathrm{d}\phi^2 + \sin^2 r\sin^2\gamma(1-\sin^2 r\sin^2\gamma)\,\mathrm{d}\varphi^2$$
$$+\tfrac{1}{2}\sin^2 2r\sin^2\gamma\,\mathrm{d}\phi\,\mathrm{d}\varphi \tag{A.45}$$
$$\mathrm{d}\mathcal{S} = \tfrac{1}{2}\sin^2(r)(\mathrm{d}\alpha_1 + \sin^2(\gamma)\,\mathrm{d}\alpha_2).$$

In [16] Sudarshan, Anandan and Govindarajan give an equivalent expression to (A.43). It must be noted though that the expression derived here for an infinitesimal triangle starts from an *exact* expression (essentially (4.17) when written in terms of state vector components) which is rather simple and that happens to be just a trigonometric equation of $\mathcal{P}$.

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